Math 127A, Fall 2019.

HW 9 Solutions (mostly adapted from Abbott's Instructor's Manual)

4.2.2 (a). As $|(5x-6)-9| < \epsilon$ is equivalent to $5|x-3| < \epsilon$ and then to $5|x-3| < \epsilon/5$, the largest δ is $\delta = \epsilon/5 = 1/5$.

4.2.5 (c). Pick an $\epsilon > 0$. We need to produce a δ so that $|x-2| < \delta$ implies $|(x^2+x-1)-5| < \epsilon$. The latter is equivalent to $|x^2+x-6| < \epsilon$, then to $|(x+3)(x-2)| < \epsilon$, then to $|x+3||x-2| < \epsilon$. Assume $|x-2| < 1$. Under this assumption, $-1 < x-2 < 1$ and so $4 < x+3 < 6$, and so $|x+3| < 6$; therefore, under this assumption, $|x+3||x-2| < \epsilon$ is implied by $6|x-2| < \epsilon$, that is, by $|x-2| < \epsilon/6$. We may take $\delta = \min\{1, \epsilon/6\}$.

4.2.6. (a) True, by definition of the limit.
(b) False. The value of $\lim_{x\to a} f(x)$ is unaffected by the value $f(a)$, so we can declare that, say, $f(a) = L+1$, without changing the fact that $\lim_{x\to a} f(x) = L$.
(c) True, by the algebraic theorem.
(d) False. Let $A = \mathbb{R} \setminus \{0\}$, and let $f, g : A \to \mathbb{R}$ be given by $f(x) = x$ and $g(x) = 1/x$, so that $f(x)g(x) = 1$ for every $x \in A$. Then $\lim_{x\to 0} f(x) = 0$, but $\lim_{x\to 0} f(x)g(x) = 1$.

4.2.7. There exists an $M > 0$ so that $|f(x)| \le M$ for every $x \in A$. Assume a is an accumulation point of A. Assume $x_n \in A$ satisfied $x_n \ne a$ and $x_n \to a$. Then

$$|f(x_n)g(x_n)| \le Mg(x_n)$$

and we know that $g(x_n) \to 0$, therefore by the order theorem for sequences, $f(x_n)g(x_n) \to 0$.

4.2.11. Assume a is an accumulation point of A. Assume $x_n \in A$ satisfies $x_n \ne a$ and $x_n \to a$. Then $f(x_n) \to L$, $h(x_n) \to L$ and $f(x_n) \le g(x_n) \le h(x_n)$, therefore by the sandwich theorem for sequences, $g(x_n) \to L$.

4.3.1 Fix an $\epsilon > 0$. We need to find a $\delta > 0$ so that $|x| < \delta$ implies

$$|\sqrt[3]{x} - \sqrt[3]{c}| < \epsilon.$$

(a) When $c = 0$, the inequality in the display is equivalent to $|x| < \epsilon^3$, so we can take $\delta = \epsilon^3$.
When $c \ne 0$, the inequality in the display is equivalent to

$$\frac{|x-c|}{|x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}|} < \epsilon.$$

Assume $|x-c| < |c|$, so that x and c have the same sign. Then all terms inside the absolute value in the denominator are positive, and so the denominator is at least $c^{2/3}$. Therefore, we may take $\delta = \min\{|c|, \epsilon \cdot c^{2/3}\}$.

4.3.6. (a) Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases} \qquad g(x) = \begin{cases} 0 & x > 0 \\ 1 & x \le 0 \end{cases}$$

Then $f(x) + g(x) = 1$ and $f(x)g(x) = 0$ for every x, but neither f nor g are continuous at 0.
(b) Impossible: $g(x) = (f(x) + g(x)) - g(x)$, so if $f + g$ and f are continuous at 0, so is g by the algebraic theorem.
(e) Impossible: if $h(x) = f(x)^3$ and $g(x) = x^{1/3}$, then h is continuous at 0, and g is continuous at $h(0) = f(0)^3$ (as it continuous on $\mathbb{R}$ by problem 4.3.1), and so by the composition theorem, $f = g_\circ h$ is continuous at 0.

4.3.8. (a) True. Pick $x_n < 1$ so that $x_n \to 1$. Then $f(x_n) \geq 0$ and $f(x_n) \to f(1)$, so the statement follows from the order theorem for sequences.
(b) True. Take a sequence of rational numbers $r_n \to x$. Then $0 = f(r_n) \to f(x)$ and so $f(x) = 0$.
(c) True. Take $\epsilon = g(x_0)/2 > 0$. There exists a $\delta > 0$ so that $|g(x) - g(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. Therefore, on $(x_0 - \delta, x_0 + \delta)$, $g(x) \geq |g(x_0)| - |g(x) - g(x_0)| > g(x_0) - \epsilon = \epsilon$. It follows that $g(x) > 0$ on $(x_0 - \delta, x_0 + \delta)$, which is an interval of strictly positive length and thus uncountable.