Math 25, Fall 2014.
Oct 31, 2014.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ________________

ID#: ________________

Instructions: Each of the ~~5~~ 4 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work for full credit. Carefully prove each assertion you make unless explicitly instructed otherwise.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. (a) Assume (a_n) is a sequence of real numbers, and $a \in \mathbb{R}$. Define precisely what this statement means: $\lim_{n\to\infty} a_n = a$.

$$(\forall \varepsilon > 0)\ (\exists N \in \mathbb{N})\ (\forall n \geq N)(|a_n - a| < \varepsilon)$$

(b) Assume (a_n) is a sequence of real numbers. Define precisely what this statement means: (a_n) is bounded.

$$(\exists M \geq 0)\ (\forall n \in \mathbb{N})\ (|a_n| \leq M)$$

(c) Let $a_n = \dfrac{2\sqrt{n}+1}{2\sqrt{n}-3}$. Using only the definition in (a), prove that that $\lim_{n\to\infty} a_n = a$ for some $a \in \mathbb{R}$. (Identify a first.)

$a = 1$. Fix an $\varepsilon > 0$. We need to find an N so that $n \geq N$ will guarantee than $|a_n - 1| < \varepsilon$.

$$\left|\frac{2\sqrt{n}+1}{2\sqrt{n}-3} - 1\right| < \varepsilon$$

$$\Updownarrow$$

$$\left|\frac{4}{2\sqrt{n}-3}\right| < \varepsilon$$

$n \geq 4$

$$\Updownarrow$$

$$\frac{4}{2\sqrt{n}-3} < \varepsilon$$

$$\Updownarrow$$

$$\frac{4}{\varepsilon} < 2\sqrt{n} - 3 \Leftrightarrow 2\sqrt{n} > \frac{4}{\varepsilon} - 3 \Leftarrow \underline{n > \left(\frac{2}{\varepsilon}\right)^2}$$

Take $N = \left\lfloor \left(\frac{2}{\varepsilon}\right)^2 \right\rfloor + 1$.

(d) Is the sequence (a_n) from (c) bounded?

Yes, every convergent sequence is bounded.

2. (a) Recall that $n! = 1 \cdot 2 \cdots n$ is the product of first n natural numbers. Prove by induction that for all $n \in \mathbb{N}$, $n \geq 4$ implies that $2^n < 3(n-1)!$.

$(n=4)$ $\quad 2^4 < 3 \cdot 3!$, $\quad 16 < 18$ ✓

$(n \to n+1)$ Assume $2^n < 3(n-1)!$ and $n \geq 4$. Then

$$2^{n+1} = 2 \cdot 2^n < 2 \cdot 3(n-1)! \leq n \cdot 3(n-1)! = 3n!$$
$$= 3((n+1)-1)!$$

So $\quad 2^{n+1} < 3((n+1)-1)!$

(b) Use (a), or any other method, to prove that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

By (a): $\quad 0 < \frac{2^n}{n!} < \frac{3(n-1)!}{n!} = \frac{3}{n}$

As $\frac{3}{n} \to 0$ as $n \to \infty$, the same must be true for $\frac{2^n}{n!}$ (by order theorems)

3. Find

$$\bigcap_{n=1}^{\infty}\left(1-\frac{1}{n}, 2+\frac{1}{n}\right).$$

Carefully prove your assertion.

This intersection equals $[1,2]$.

Clearly $x \in [1,2]$ implies (that $x \in (1-\frac{1}{n}, 2+\frac{1}{n})$ for $\forall n \in \mathbb{N}$, so that $\underline{[1,2] \subseteq \bigcap_{n=1}^{\infty} (1-\frac{1}{n}, 2+\frac{1}{n})}$.

We now show that $x \notin [1,2]$ implies $x \notin \bigcap_{n=1}^{\infty} (1-\frac{1}{n}, 2+\frac{1}{n})$. We look at two cases:

- If $x > 2$, $\frac{1}{n} < x-2$ if n is large enough, and so $2+\frac{1}{n} < x$ for large enough n, $x \notin (1-\frac{1}{n}, 2+\frac{1}{n})$.
- If $x < 1$, $\frac{1}{n} < 1-x$ if n is large enough, and so $x < 1-\frac{1}{n}$, and $x \notin (1-\frac{1}{n}, 2+\frac{1}{n})$.

In either case, there exists an n so that $x \notin (1-\frac{1}{n}, 2+\frac{1}{n})$, so $x \notin \bigcap_{n=1}^{\infty} (1-\frac{1}{n}, 2+\frac{1}{n})$.

This proves that $\underline{\bigcap_{n=1}^{\infty} (1-\frac{1}{n}, 2+\frac{1}{n}) \subseteq [1,2]}$.

The two inclusions show that

$$\bigcap_{n=1}^{\infty} \left(1-\frac{1}{n}, 2+\frac{1}{n}\right) = [1,2].$$

4. (a) State the definition of $\sup A$ for a set $A \subset \mathbb{R}$.

$b = \sup A$ if

(i) b is an upper bound for A, i.e., $a \leq b$ for every $a \in A$; and

(ii) for every upper bound c for A, $b \leq c$.

(b) What do you need to assume about a set $A \subset \mathbb{R}$ to ensure that $\sup A$ exists?

A has to be bounded above ($\exists\, b \in \mathbb{R}$ so that $a \leq b$ for every $a \in A$), and $A \neq \emptyset$.

(c) Let $A = (0, 2)$ and $B = A \cap \mathbb{Q}$. Find $\sup A$ and $\sup B$. Carefully prove your assertions.

$\sup A = \sup B = 2$.

Clearly, $\sup A \leq 2$ and $\sup B \leq 2$.

We claim that $\forall \varepsilon > 0$, $\exists\, r \in B$ so that $r > 2 - \varepsilon$. This is true as there exists an $r \in \mathbb{Q}$ in the interval $(2 - \varepsilon, 2)$ by density of rational numbers. This implies that $\sup B = 2$ and that $\sup A = 2$ (as such r is also in A).

(d) Assume that $A \subseteq \mathbb{R}$. Assume also that A is bounded above and that $A \cap \mathbb{Q}$ is nonempty. Is it necessarily true that $\sup A = \sup(A \cap \mathbb{Q})$? ~~Carefully prove your assertion.~~

No. Take $A = (0, 2) \cup \{1 + \sqrt{2}\}$.

Then $A \cap \mathbb{Q} = (0, 2) \cap \mathbb{Q}$ as $1 + \sqrt{2} \notin \mathbb{Q}$.

$\sup A = \max A = 1 + \sqrt{2}$

$\sup(A \cap \mathbb{Q}) = 2$