Math 127A, Fall 2019.
Oct. 25, 2019.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work for full credit. Carefully prove each assertion you make unless explicitly instructed otherwise.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. (a) Assume (a_n) is a sequence of real numbers, and $a \in \mathbb{R}$. Define precisely what this statement means: $\lim_{n\to\infty} a_n = a$.

$$(\forall \varepsilon > 0)\ (\exists N \in \mathbb{N})\ (\forall n \geq N)\ (|a_n - a| < \varepsilon)$$

(b) Let $a_n = \dfrac{3n^2+1}{n^2+3}$. Using only the definition in (a), prove that that $\lim_{n\to\infty} a_n = a$ for some $a \in \mathbb{R}$. (Identify a first.)

$$a_n = \frac{3 + \frac{1}{n^2}}{1 + \frac{3}{n^2}} \longrightarrow 3 \quad \text{by algebraic limit thm.}$$

$$|a_n - 3| = \left| \frac{3n^2+1}{n^2+3} - 3 \right| = \left| \frac{\cancel{3n^2} + 1 - \cancel{3n^2} - 9}{n^2+3} \right| = \frac{8}{n^2+3}$$

$$|a_n - 3| < \varepsilon \iff \frac{8}{n^2+3} < \varepsilon \Leftarrow \frac{8}{n^2} < \varepsilon \iff \frac{8}{\varepsilon} < n^2$$

We may take N to be any integer strictly larger than $\sqrt{\frac{8}{\varepsilon}}$; for example, $N = \left\lfloor \sqrt{\frac{8}{\varepsilon}} \right\rfloor + 1$.

(c) Let $b_n = (-1)^n a_n$, where (a_n) is the sequence from (b). Compute $\limsup_{n\to\infty} b_n$ and $\liminf_{n\to\infty} b_n$.

If n is even, $b_n = a_n \longrightarrow 3$.

If n is odd, $b_n = -a_n \to -3$.

So,

$\limsup a_n = 3$

$\liminf a_n = -3$.

2. *Do not consider extended real line in this problem; i.e., no $+\infty$ or $-\infty$.*

(a) State the definition of $\sup A$ for a set $A \subset \mathbb{R}$.

$S = \sup A \in \mathbb{R}$ if

(i) s is an upper bound for A, i.e., $x \le s$ for $\forall x \in A$

(ii) for every upper bound b for A, $b \le s$.

(b) Determine

$$A = \bigcup_{n=1}^{\infty} \left[-\frac{1}{n}, 1 - \frac{1}{n}\right). = [-1, 1)$$

Give the answer as an interval.

Proof. For all $n \in \mathbb{N}$, $[-\frac{1}{n}, 1-\frac{1}{n}) \subseteq [-1, 1)$, so $A \subseteq [-1, 1)$. Moreover, $[-1, 0) = [-\frac{1}{1}, 1-\frac{1}{1}) \subseteq A$. Also, $0 \in [-\frac{1}{2}, 1-\frac{1}{2}) = [-\frac{1}{2}, \frac{1}{2}) \subseteq A$. For any $x \in (0, 1)$, there exists an $n \in \mathbb{N}$, so that $1 - x > \frac{1}{n}$, and so $-\frac{1}{n} < 0 < x < 1 - \frac{1}{n}$, and $x \in [\frac{1}{n}, 1-\frac{1}{n})$. So, $x \in A$. It follows that $[-1, 1) \subseteq A$.

The two inclusions prove the claim.

(c) For the set A in (b), compute $\sup A$ and $\inf A$.

$\min A = \inf A = -1$, as $x \geq -1$ for all $x \in A$.

$\sup A = 1$: (i) 1 is an upper bound, as $x \le 1$ for all $x \in A$

(ii) $1 - \varepsilon$ can not be an upper bd. for any $\varepsilon > 0$, as there exists an $x \in \mathbb{R}$ such that $1 - \varepsilon < x < 1$, so $x \in A$ but $x > 1 - \varepsilon$.

So 1 is the least upper bd.

3. In each of the parts (b)–(d), your answer should be supported by a proof or a counterexample.
(a) Assume (a_n) is a sequence of real numbers. Define precisely what this statement means: (a_n) is bounded.

$$(\exists M \in \mathbb{R})(\forall n \in \mathbb{N})(|a_n| \leq M)$$

(b) True or false: every bounded sequence (a_n) is convergent.

No. $a_n = (-1)^n = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$ does not converge, as $\limsup a_n = 1$ and $\liminf a_n = -1$ are not equal. But $|a_n| = 1$ so a_n is bounded.

(c) True or false: every bounded sequence (a_n) has a convergent subsequence.

Yes. This is the Bolzano-Weierstrass thm.

(d) True or false: if (a_n) is bounded, then $\lim_{n\to\infty} \frac{a_n}{n} = 0$.

Yes. Find an $M \geq 0$ so that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Then

$$\left|\frac{a_n}{n}\right| \leq \frac{M}{n}, \quad \text{so } -\frac{M}{n} \leq \frac{a_n}{n} \leq \frac{M}{n}$$

and $\frac{a_n}{n} \to 0$ by the sandwich theorem.

4. Assume the sequence (x_n) is given recursively by $a_1 = 1$ and

$$a_{n+1} = \frac{1}{2}a_n + \sqrt{a_n} \qquad \text{for } n = 1, 2, \ldots.$$

Show that the sequence is increasing, that $a_n \le 4$ for all $n \in \mathbb{N}$, and that $\lim_{n\to\infty} a_n$ exists. Compute the limit.

Step 1. $0 \le a_n \le 4$ for all $n \in \mathbb{N}$,

$(n=1)$ $a_1 = 1$ ✓

$(n \to n+1)$ If $a_n \le 4$, $a_{n+1} \le \frac{1}{2} \cdot 4 + \sqrt{4} = 4$.

Step 2. $a_n \le a_{n+1}$ for all $n \ge 1$.

$(n=1)$ $a_1 = 1$, $a_2 = \frac{1}{2} + \sqrt{1} = \frac{3}{2}$, so $a_1 < a_2$.

$(n \to n+1)$ If $a_n \le a_{n+1}$, $\frac{1}{2}a_n \le \frac{1}{2}a_{n+1}$

and $\sqrt{a_n} \le \sqrt{a_{n+1}}$, so

$$\underbrace{\frac{1}{2}a_n + \sqrt{a_n}}_{= a_{n+1}} \le \underbrace{\frac{1}{2}a_{n+1} + \sqrt{a_{n+1}}}_{= a_{n+2}}$$

$$\Downarrow$$

$$a_{n+1} \le a_{n+2}$$

As the sequence is increasing and bounded, it converges. Let $a = \lim a_n$. Then

$a = \frac{1}{2}a + \sqrt{a}$, by algebraic laws

$a = 2\sqrt{a}$

$a^2 = 4a$, $a = 0$ or 4

As $a = 0$ is impossible $(a \ge a_1 = 1)$,

$\underline{\underline{a = 4}}$.