Math 25, Fall 2014.
Nov. 21, 2014.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____________________

ID#: ____________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work for full credit. Carefully prove each assertion you make unless explicitly instructed otherwise.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. Assume $x_1 = 1$ and

$$x_{n+1} = \sqrt{6 + x_n} \qquad \text{for } n \in \mathbb{N}.$$

Show that the sequence is increasing, and that $\lim_{n\to\infty} x_n$ exists. Compute the limit.

Prove by induction that $x_n < x_{n+1}$ for every $n \in \mathbb{N}$, that $x_n \le 3$ for all $n \in \mathbb{N}$,

$(n=1)$: $x_2 = \sqrt{7} > 1 = x_1$.

$(n \to n+1)$: By the induction hypothesis, $x_n < x_{n+1}$, then $6 + x_n < 6 + x_{n+1}$ and $\sqrt{6+x_n} < \sqrt{6+x_{n+1}}$, i.e., $x_{n+1} < x_{n+2}$.

Thus, (x_n) is increasing. We need to prove that it is bounded.

Claim: $x_n \le 3$ for every n.

$(n=1)$ True $x_1 = 1 \le 3$.

$(n \to n+1)$ If $x_n \le 3$, then $x_{n+1} \le \sqrt{6+3} = 3$.

The claim is proved.

As the sequence is monotone and bounded ($0 \le x_n \le 3$ for all n), the limit $x = \lim x_n$ exists and satisfies

$$x = \sqrt{6+x}$$

$$x^2 = 6 + x$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

As $x \geq 0$, $\underline{\underline{x = 3}}$.

$\lim_{n\to\infty} x_n = 3$.

2. For part (a), you may use without definition the concept of *limit of a sequence*. Then, for each of the series in (b), (c), (d), determine (with proof) whether it converges absolutely, converges conditionally, or diverges.

(a) Assume a_n, $n \in \mathbb{N}$, are real numbers. Define precisely what these two statement mean: $\sum_{k=1}^{\infty} a_k$ converges absolutely; $\sum_{k=1}^{\infty} a_k$ converges conditionally.

$\sum_{k=1}^{\infty} a_k$ converges absolutely: $\sum_{k=1}^{n} |a_k|$ converges

$\sum_{k=1}^{\infty} a_k$ conv. conditionally: $\sum_{k=1}^{n} a_k$ converges, but $\sum_{k=1}^{n} |a_k|$ does not.

(b) $\sum_{k=1}^{\infty}\left(\frac{1}{k}+\frac{7}{k!}\right)$

As $\frac{1}{k}+\frac{1}{k!} > \frac{1}{k}$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (harmonic series), this series diverges.

(c) $\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{3k}{4k+1}\right)^{k}$

As $\sqrt[k]{|a_k|} = \frac{3k}{4k+1} \to \frac{3}{4} < 1$ as $k \to \infty$, the series converges absolutely by the root test.

(d) $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt{k}}$

As $\frac{1}{\sqrt{k}}$ is ~~dec~~ decreases and converges to 0 as $k \to \infty$, the series converges by the alternating series test.

As $\sum_{k=1}^{\infty} \left|(-1)^{k+1}\frac{1}{\sqrt{k}}\right| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges (p-series with $p = \frac{1}{2} < 1$), the series converges conditionally.

3. Assume that $a_k > 0$ for all $k \in \mathbb{N}$. For each statement below, prove it or find a counterexample.

(a) If $a_k = 1$ for all even k, then $\sum_{k=1}^{\infty} a_k$ diverges.

Yes, $a_k \not\to 0$ as $k \to \infty$, so the series diverges by the n-th term test.

(b) If $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} (a_k + a_k^2)$ converges.

We know that $\lim_{k\to\infty} a_k = 0$, and so

$$\lim_{k\to\infty} \frac{a_k + a_k^2}{a_k} = \lim_{k\to\infty} (1 + a_k) = 1,$$

so the series $\sum_{k=1}^{\infty} (a_k + a_k^2)$ converges by the limit comparison test. Yes.

(c) If $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges, then $\sum_{k=1}^{\infty} a_k^2$ converges.

No. Take $a_k = \frac{1}{\sqrt{k}}$. We know that $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k}}$ converges by 2(d), but $\sum_{k=1}^{\infty} a_k^2 = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges (as it is the harmonic series).

4. Assume (a_n) is a sequence of real numbers.
(a) Is the following statement true: if $0 \leq a_n \leq 7$ for all $n \geq 10$, then (a_n) has a convergent subsequence? (Justify your assertion.)

Yes. The sequence is bounded so the conclusion follows from Bolzano-Weierstrass theorem.

(b) Prove: if $\liminf(na_n) = 2$, then $\sum_{n=1}^{\infty} a_n$ is a divergent series.

If $\liminf(na_n) = 2$, then there exists an $N \in \mathbb{N}$ so that $na_n \geq 1$ for $n \geq N$. Then $a_n \geq \frac{1}{n}$ for $n \geq N$ and so $\sum a_n$ diverges by comparison with the harmonic series.