Math 127A, Fall 2019.
Nov. 22, 2019.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work for full credit. Carefully prove each assertion you make unless explicitly instructed otherwise.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. For part (a), you may use without definition the concept of *limit of a sequence*. Then, for each of the series in (b), (c), (d), determine (with proof) whether it converges absolutely, converges conditionally, or diverges.

(a) Assume a_n, $n \in \mathbb{N}$, are real numbers. Define precisely what these two statements mean: $\sum_{k=1}^{\infty} a_k$ converges absolutely; $\sum_{k=1}^{\infty} a_k$ converges conditionally.

$\sum_{k=1}^{\infty} a_k$ converges absolutely : $\sum_{k=1}^{n} |a_k|$ converges.

$\sum_{k=1}^{\infty} a_k$ converges conditionally : $\sum_{k=1}^{n} a_k$ converges, $\sum_{k=1}^{n} |a_k|$ diverges

(b) $\sum_{k=1}^{\infty} \underbrace{(-1)^{k+1} \frac{7^k}{\sqrt{k!}}}_{a_k}$

Ratio test :

$$\left| \frac{a_{k+1}}{a_k} \right| = \frac{7^{k+1}}{\sqrt{(k+1)!}} \frac{\sqrt{k!}}{7^k} = \frac{7}{\sqrt{k+1}} \xrightarrow[k \to \infty]{} 0$$

The series converges absolutely.

(c) $\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{1}{k} + \frac{1}{\sqrt{k}} \right)$

Alternating series test : $\frac{1}{k} + \frac{1}{\sqrt{k}}$ decreases and converges to 0, so the series converges.

$$\sum_{k=1}^{\infty} \left| (-1)^{k+1} \left(\frac{1}{k} + \frac{1}{\sqrt{k}} \right) \right| = \sum_{k=1}^{\infty} \left(\frac{1}{k} + \frac{1}{\sqrt{k}} \right) \text{ diverges}$$

as $\frac{1}{k} + \frac{1}{\sqrt{k}} \geq \frac{1}{k}$, and the harmonic series diverges.

The series converges conditionally.

(d) $\sum_{k=1}^{\infty} \underbrace{\frac{k + \sqrt{k}}{3k^2 + 4k}}_{a_k}$

Limit comparison test with $\sum_{k=1}^{\infty} \underbrace{\frac{1}{k}}_{b_k}$,

$$\frac{a_k}{b_k} = \frac{k(k + \sqrt{k})}{3k^2 + 4k} \longrightarrow \frac{1}{3}$$

As $\sum b_k$ diverges, the series $\sum a_k$ diverges.

2. Assume that $a_k > 0$ for all $k \in \mathbb{N}$. For each statement below, determine whether it is true or false. Each answer should be supported by a proof or a counterexample.

(a) If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} \frac{1}{1+a_k} = 1$.

Yes, $\lim a_k = 0$ (by the n'th term tes)

and so $\lim_{n\to\infty} \frac{1}{1+a_k} = \frac{1}{1+0} = 1.$

(b) If $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} (a_k + \sqrt{a_k})$ converges.

No, For example, $a_k = \frac{1}{k^2}$. Then $\sum_1 a_k^2 = \sum_1 \frac{1}{k^2}$ converges. Also $a_k + \sqrt{a_k} = \frac{1}{k^2} + \frac{1}{k} \geq \frac{1}{k}$, so $\sum_1 a_k + \sqrt{a_k}$ diverges by comparison with harmonic series.

(c) If $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} (-1)^{k+1} a_k^2$ converges.

Yes. In fact $\sum_{k=1}^{\infty} (-1)^{k+1} a_k^2$ converges absolutely!

As $a_k \to 0$, $\exists N$ so that $a_k \leq 1$ for $k \geq N$.

Then

$$|(-1)^{k+1} a_k^2| = a_k^2 \leq a_k \text{ for } k \geq N$$

and so $\sum_1 |(-1)^{k+1} a_k^2|$ converges by comparison with $\sum_1 a_k$.

3. (a) Assume $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Define precisely what this statement means: x is an accumulation point of A.

$$(\forall \varepsilon > 0)\ (A \cap (V_\varepsilon(x) \setminus \{x\}) \neq \emptyset)$$

(b) Assume $A \subseteq \mathbb{R}$. Define precisely what this statement means: A is closed.

Every limit point of A is in A.

(c) Let $A = (1,2) \cup (3,4]$ Is A open? Is A closed? Determine its closure $\overline{A}$.

Not open: $4 \in A$, but no neighborhood of 4 is entirely included in A

Not closed: 1 is a limit point of A, but is not in A.

$$\overline{A} = [1,2] \cup [3,4]$$

(d) Let $A = [-1,1] \cap \mathbb{Q}$. Determine its interior A° and its closure $\overline{A}$. Is the set A compact?

$A^\circ = \emptyset$: For every $x \in [-1,1] \cap \mathbb{Q}$, and every $\varepsilon > 0$, $V_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$ contains an irrational point, so $V_\varepsilon(x) \not\subseteq A$,

$\overline{A} = [-1,1]$: For every $x \in [-1,1]$, $V_\varepsilon(x)$ contains a rational point in $[-1,1]$.

A is not compact: $A \neq \overline{A}$ and so A is not closed

4. Assume $A \subseteq \mathbb{R}$. For each statement below, determine whether it is true or false. Each answer should be supported by a proof or a counterexample.

(a) If A^c is open and A is bounded, then A is compact.

Yes. As A^c is open, A is closed. A closed and bounded set is compact.

(b) If $x \in (0,1)$ and x an interior point of A, then x is an interior point of $A \cap (0,1)$.

Yes. As $(0,1)$ is open, $\exists\, \varepsilon_1 > 0$ so that $V_{\varepsilon_1}(x) \subseteq (0,1)$

As A is open, $\exists\, \varepsilon_2 > 0$ so that $V_{\varepsilon_2}(x) \subseteq A$.

Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\} > 0$. Then $V_\varepsilon(x) \subseteq V_{\varepsilon_1}(x) \cap V_{\varepsilon_2}(x)$
$\subseteq (0,1) \cap A$.

(c) If A is compact, then $A \cup [0,1] \cup \{2\}$ is compact.

Yes, A is bounded, and so are $[0,1]$ and $\{2\}$. So $A \cup [0,1] \cup \{2\}$ is bounded, as a union of 3 bounded sets. Also, A, $[0,1]$ and $\{2\}$ are all closed, and so $A \cup [0,1] \cup \{2\}$ is closed as a union of 3 closed sets.

(d) If A is disconnected, then its closure $\overline{A}$ is disconnected.

No. $A = (-\infty, 0) \cup (0, \infty)$ is disconnected, but $\overline{A} = \mathbb{R}$ is connected.