Homework 7, due Thu., Dec. 7

1. Joint density of \((X,Y)\) is given by
   \[ f(x,y) = x e^{-x(y+1)}, \quad x, y > 0. \]
   (a) Find the conditional density of \(Y\) given \(X = x\). (b) Compute the density of \(Z = XY\).

2. Assume that \(X_1, \ldots, X_{10}\) are independent \(\text{Exponential}(\lambda)\) random variables. Compute the densities of \(H = \max\{X_1, \ldots, X_{10}\}\) and \(L = \min\{X_1, \ldots, X_{10}\}\).

3. Select a point \((X,Y)\) at random from the square \([-1,1] \times [-1,1]\). Compute (a) \(E(|X| + |Y|)\), (b) \(E|XY|\), and (c) \(E|X - Y|\).

4. Suppose that Alice and Bob, each randomly and independently, choose 3 out of 10 objects without replacement. Find the expected number of objects that are (a) chosen by both Alice and Bob, (b) not chosen by either Alice or Bob, (c) chosen by exactly one of them.

5. A coin has probability \(p\) of Heads. Toss this coin \(n\) times, and let \(X\) be the number of tosses, from toss 2 on, that have different outcome than the previous toss. Compute \(E_X\).

6. Let \((X,Y)\) be a random point of in the square \(\{(x,y) : 0 \leq x, y \leq 1\}\). (a) Compute the density of \(Z = XY\), \(EZ\), and \(\text{Var}(Z)\). (b) Assume that 1000 such points \((X_i,Y_i)\), \(i = 1, \ldots, 1000\), are chosen independently, and approximate the probability \(P(X_1Y_1 + X_2Y_2 + \cdots + X_{1000}Y_{1000} < 255)\).

You should also do the six Problems in Section 8 of the book.
1. (a) After integration, \( f_X(x) = e^{-x} \) for \( x > 0 \), and so 
\[
 f_Y(y|X = x) = x e^{-xy},
\]
for \( x, y > 0 \), Exponential\((x)\).

(b) For \( z \geq 0 \), \( P(Z \leq z) = P(Y \leq z/X) = \int_0^{\infty} dx \int_0^{z/x} f(x, y) dy = 1 - e^{-z} \), so that \( f_Z(z) = e^{-z} \), Exponential\((1)\).

2. \( P(H \leq h) = P(X_1 \leq h) = 10(1 - e^{-\lambda h})^{10} \) and so 
\[
 f_H(h) = 10\lambda e^{-\lambda h}(1 - e^{-\lambda h})^9.
\]
Moreover, \( P(L \leq \ell) = 1 - P(L \geq \ell) = 1 - e^{-10\lambda \ell} \), and so 
\[
 f_L(\ell) = 10\lambda e^{-10\lambda \ell},
\]
so \( L \) is Exponential\((10\lambda)\).

3. (a) \(|X|\) is Uniform on \([0, 1]\), so \( E|X| = \frac{1}{2} \), and the answer is 1. (b) By independence, the answer is \( \frac{1}{4} \). (c) \( \frac{1}{2} \int_{-1}^{1} dx \int_{-1}^{x-y} (x-y) dy = \frac{1}{4} \).

4. By the indicator trick: (a) \( 10(0.3)^2 \), (b) \( 10(0.7)^2 \), (c) \( 10 \cdot 2 \cdot 0.3 \cdot 0.7 \).

5. By the indicator trick: \( (n - 1) \cdot 2p(1-p) \).

6. (a) For \( z \in (0, 1) \), \( P(Z \leq z) = P(XY \leq z) = 1 - \int_0^z (1 - \frac{z}{x}) dx = z - z \ln z \) and so \( f_Z(z) = -\ln z \). By independence \( E(XY) = EX \cdot EY = 1/4 \) and \( E((XY)^2) = EX^2 \cdot EY^2 = 1/9 \), so \( EZ = 1/4 \) and \( \text{Var}(Z) = 7/144 \).

(b) Let \( S \) be the sum. Then \( ES = 250 \) and \( \text{Var}(S) = 875/18 \), so 
\[
 P(S \leq 255) = P \left( \frac{S - 250}{\sqrt{875/18}} \leq \frac{5}{\sqrt{875/18}} \right) \approx \Phi \left( \frac{3}{\sqrt{17.5}} \right) \approx 0.7634.
\]