A Geometric($p$) random variable $X$ counts the number of trials required for the first success in independent trials with success probability $p$.

**Properties:**

(1) Probability mass function: First of all notice that we need at least one trial to get the first success, therefore the lowest value of $X$ is 1. And we may need 100, 234, 10000000, ... etc. trials to get the first success, therefore there is no maximum value of $X$ (unlike Binomial distribution). So $X$ can take values 1, 2, 3, ... i.e., any positive integer.

We want to compute $P(X = n)$. In other words we want to compute the probability that we have ($n - 1$) “failure”s in first ($n - 1$) trials and the first “success” occurs at $n$th trial. Obviously by the independence of trials we have $P(X = n) = (1 - p)^{n-1}p$.

(2) Computation of $E[X]$: By the definition of expectation we have

$$E[X] = \sum_{n=1}^{\infty} nP(X = n)$$

$$= \sum_{n=1}^{\infty} n(1-p)^{n-1}p$$

$$= p \sum_{n=1}^{\infty} n(1-p)^{n-1}$$

$$= pS,$$  \hspace{1cm} (1)

where

$$S := \sum_{n=1}^{\infty} n(1-p)^{n-1}.$$  \hspace{1cm} (3)

To compute $S$ we notice that

$$S - (1-p)S = \sum_{n=1}^{\infty} n(1-p)^{n-1} - \sum_{n=1}^{\infty} n(1-p)^n$$

$$= \sum_{m=0}^{\infty} (m + 1)(1-p)^m - \sum_{n=1}^{\infty} n(1-p)^n \quad \text{(taking } n - 1 = m \text{ in the first sum)}$$

$$= 1 + \sum_{m=1}^{\infty} (m + 1)(1-p)^m - \sum_{n=1}^{\infty} n(1-p)^n$$

$$= 1 + \sum_{m=1}^{\infty} (m + 1)(1-p)^m - \sum_{m=1}^{\infty} m(1-p)^m \quad \text{(renaming } n = m \text{ in second sum)}$$

$$1$$
\[
\begin{align*}
1 + \sum_{m=1}^{\infty} (1 - p)^m \\
= 1 + \frac{1 - p}{1 - (1 - p)} \text{ (sum of geometric series)} \\
= 1 + \frac{1 - p}{p} \\
= \frac{1}{p}.
\end{align*}
\]

Therefore we have \( S - (1 - p)S = \frac{1}{p} \), i.e., \( S = \frac{1}{p^2} \). Consequently we have

\[ E[X] = pS = \frac{1}{p}. \]

**Alternative Method:** Define a function

\[ f(p) = \sum_{n=1}^{\infty} (1 - p)^n \quad (5) \]

\[ = \frac{1 - p}{1 - (1 - p)} \\
= \frac{1 - p}{p} - 1 \quad (6) \]

Differentiating the function \( f \) with respect to \( p \) we obtain (using the definition of \( f \) given by \( (5) \))

\[ f'(p) = -\sum_{n=1}^{\infty} n(1 - p)^{n-1} \\
= -S \quad (S \text{ is defined in } (3)) \]

On the other hand differentiating \( (6) \) with respect to \( p \) we obtain

\[ f'(p) = -\frac{1}{p^2}. \]

Comparing the above two we have

\[ -S = -\frac{1}{p^2} \]

\[ i.e., \quad S = \frac{1}{p^2}. \quad (7) \]

Consequently, using \( (2) \), \( (3) \), and \( (7) \) we get

\[ E[X] = pS = \frac{1}{p^2} = \frac{1}{p}. \]

(3) Computation of \( \text{Var}(X) \): From the definition of variance we have

\[ \text{Var}(X) = E[X^2] - \{E[X]\}^2. \]
To compute $\mathbb{E}[X^2]$,

\[
\mathbb{E}[X^2] = \sum_{n=1}^{\infty} n^2 P(X = n) = \sum_{n=1}^{\infty} n^2 (1-p)^{n-1} p = p \sum_{n=1}^{\infty} n^2 (1-p)^{n-1} = p T, \tag{8}
\]

where

\[
T = \sum_{n=1}^{\infty} n^2 (1-p)^{n-1}.
\]

Now we want to compute the value of $T$. Proceeding as (4),

\[
T - (1-p)T = \sum_{n=1}^{\infty} n^2 (1-p)^{n-1} - \sum_{n=1}^{\infty} n^2 (1-p)^n
= \sum_{m=0}^{\infty} (m+1)^2 (1-p)^m - \sum_{n=1}^{\infty} n^2 (1-p)^n \text{ (taking } n-1 = m \text{ in the first sum)}
= 1 + \sum_{m=1}^{\infty} (m+1)^2 (1-p)^m - \sum_{m=1}^{\infty} m^2 (1-p)^m \text{ (renaming } n = m \text{ in the second sum)}
= 1 + \sum_{m=1}^{\infty} [(m+1)^2 - m^2] (1-p)^m
= 1 + \sum_{m=1}^{\infty} (2m+1)(1-p)^m
= 1 + 2 \sum_{m=1}^{\infty} m(1-p)^m + \sum_{m=1}^{\infty} (1-p)^m
= 1 + 2 \frac{1-p}{p} \sum_{m=1}^{\infty} m(1-p)^{m-1} p + \frac{1-p}{1-(1-p)}
= 1 + 2 \frac{1-p}{p} \mathbb{E}[X] + \frac{1-p}{p}
= 1 + \frac{2(1-p)}{p^2} + \frac{1-p}{p} \text{ (since } \mathbb{E}[X] = 1/p)\]

\[
= 1 + \frac{2}{p^2} - \frac{2}{p} + \frac{1}{p} - 1
= \frac{2}{p^2} - \frac{1}{p}.
\]
So we obtain \( pT = \frac{2}{p} - \frac{1}{p} \). Therefore \( \text{Var}(X) \) is given by

\[
\text{Var}(X) = \mathbb{E}[X^2] - \left( \mathbb{E}[X] \right)^2 \\
= pT - \frac{1}{p^2} \quad \text{(since } \mathbb{E}[X^2] = pT \text{ as in (8) and } \mathbb{E}[X] = \frac{1}{p}) \\
= \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} \\
= \frac{1}{p^2} - \frac{1}{p} \\
= \frac{1 - p}{p^2}.
\]

**Alternative Method:** This method is similar as the alternative method described above. Differentiating the same function \( f \) (defined in (5)) twice with respect to \( p \) we get

\[
f''(p) = \sum_{n=1}^{\infty} n(n-1)(1-p)^{n-2}
\]

On the other hand, differentiating (6) twice with respect to \( p \) we get

\[
f''(p) = \frac{2}{p^3}.
\]

Therefore we have

\[
\sum_{n=1}^{\infty} n(n-1)(1-p)^{n-2} = \frac{2}{p^3}. \quad (9)
\]

Now we notice that

\[
\mathbb{E}[X^2] = \sum_{n=1}^{\infty} n^2(1-p)^{n-1}p \\
= \sum_{n=1}^{\infty} [n(n-1) + n](1-p)^{n-1}p \\
= p(1-p) \left[ \sum_{n=1}^{\infty} n(n-1)(1-p)^{n-2} \right] + \sum_{n=1}^{\infty} n(1-p)^{n-1}p \\
= p(1-p) \frac{2}{p^2} + \mathbb{E}[X] \quad \text{(using (9))} \\
= \frac{2(1-p)}{p^2} + \frac{1}{p} \\
= \frac{2}{p^2} - \frac{1}{p}.
\]

Therefore we have

\[
\text{Var}(X) = \mathbb{E}[X^2] - \left( \mathbb{E}[X] \right)^2 \\
= \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} \\
= \frac{1}{p^2} - \frac{1}{p} \\
= \frac{1 - p}{p^2}.
\]