MAT 135A

Geometric Distribution

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A Geometric(p) random variable X counts the number of trials required for the first success in independent trials with success probability p.

Properties:

(1) Probability mass function: First of all notice that we need at least one trial to get the first success, therefore the lowest value of X is 1. And we may need $100, 234, 10000000, \cdots$ etc. trials to get the first success, therefore there is no maximum value of X (unlike Binomial distribution). So X can take values $1, 2, 3, \ldots$ i.e., any positive integer.

We want to compute $\mathbf{P}(X = n)$. In other words we want to compute the probability that we have $(n-1)$ "failure"s in first $(n-1)$ trials and the first "success" occurs at nth trial. Obviously by the independence of trials we have $\mathbb{P}(X = n) = (1-p)^{n-1}p$.

(2) Computation of $\mathbb{E}[X]$: By the definition of expectation we have

$$
\begin{aligned}
\mathbb{E}[X] &= \sum_{n=1}^{\infty} n\mathbb{P}(X = n) \\
&= \sum_{n=1}^{\infty} n(1-p)^{n-1}p \\
&= p\sum_{n=1}^{\infty} n(1-p)^{n-1} \qquad (1) \\
&= pS, \qquad (2)
\end{aligned}
$$

where

$$S := \sum_{n=1}^{\infty} n(1-p)^{n-1}. \qquad (3)$$

To compute S we notice that

$$
\begin{aligned}
S - (1-p)S &= \sum_{n=1}^{\infty} n(1-p)^{n-1} - \sum_{n=1}^{\infty} n(1-p)^{n} \qquad (4) \\
&= \sum_{m=0}^{\infty} (m+1)(1-p)^{m} - \sum_{n=1}^{\infty} n(1-p)^{n} \quad \text{(taking } n-1 = m \text{ in the first sum)} \\
&= 1 + \sum_{m=1}^{\infty} (m+1)(1-p)^{m} - \sum_{n=1}^{\infty} n(1-p)^{n} \\
&= 1 + \sum_{m=1}^{\infty} (m+1)(1-p)^{m} - \sum_{m=1}^{\infty} m(1-p)^{m} \quad \text{(renaming } n = m \text{ in second sum)}
\end{aligned}
$$

$$
\begin{aligned}
&= 1 + \sum_{m=1}^{\infty} (1-p)^m \\
&= 1 + \frac{1-p}{1-(1-p)} \quad \text{(sum of geometric series)} \\
&= 1 + \frac{1-p}{p} \\
&= \frac{1}{p}.
\end{aligned}
$$

Therefore we have $S - (1-p)S = \frac{1}{p}$ i.e., $S = \frac{1}{p^2}$. Consequently we have

$$\mathbb{E}[X] \quad = \quad pS = \frac{1}{p}.$$

<u>Alternative Method:</u> Define a function

$$
\begin{aligned}
f(p) &= \sum_{n=1}^{\infty} (1-p)^n && (5) \\
&= \frac{1-p}{1-(1-p)} \\
&= \frac{1}{p} - 1 && (6)
\end{aligned}
$$

Differentiating the function f with respect to p we obtain (using the definition of f given by (5))

$$
\begin{aligned}
f'(p) &= -\sum_{n=1}^{\infty} n(1-p)^{n-1} \\
&= -S \quad (S \text{ is defined in (3)})
\end{aligned}
$$

On the other hand differentiating (6) with respect to p we obtain

$$f'(p) = -\frac{1}{p^2}.$$

Comparing the above two we have

$$
\begin{aligned}
&-S = -\frac{1}{p^2} \\
i.e., \qquad &S = \frac{1}{p^2}. && (7)
\end{aligned}
$$

Consequently, using (2), (3), and (7) we get

$$\mathbb{E}[X] = pS = p\frac{1}{p^2} = \frac{1}{p}.$$

(3) Computation of $\mathrm{Var}(X)$: From the definition of variance we have

$$Var(X) = \mathbb{E}[X^2] - \{\mathbb{E}[X]\}^2.$$

To compute $\mathbb{E}[X^2]$,

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{n=1}^{\infty} n^2 \mathbb{P}(X=n) \\
&= \sum_{n=1}^{\infty} n^2 (1-p)^{n-1} p \\
&= p \sum_{n=1}^{\infty} n^2 (1-p)^{n-1} \\
&= pT, \qquad (8)
\end{aligned}$$

where

$$T = \sum_{n=1}^{\infty} n^2 (1-p)^{n-1}.$$

Now we want to compute the value of T. Proceeding as (4),

$$\begin{aligned}
T-(1-p)T &= \sum_{n=1}^{\infty} n^2 (1-p)^{n-1} - \sum_{n=1}^{\infty} n^2 (1-p)^n \\
&= \sum_{m=0}^{\infty} (m+1)^2 (1-p)^m - \sum_{n=1}^{\infty} n^2 (1-p)^n \quad \text{(taking } n-1=m \text{ in the first sum)} \\
&= 1 + \sum_{m=1}^{\infty} (m+1)^2 (1-p)^m - \sum_{m=1}^{\infty} m^2 (1-p)^m \quad \text{(renaming } n=m \text{ in the second sum)} \\
&= 1 + \sum_{m=1}^{\infty} [(m+1)^2 - m^2](1-p)^m \\
&= 1 + \sum_{m=1}^{\infty} (2m+1)(1-p)^m \\
&= 1 + 2\sum_{m=1}^{\infty} m(1-p)^m + \sum_{m=1}^{\infty} (1-p)^m \\
&= 1 + 2\frac{1-p}{p} \sum_{m=1}^{\infty} m(1-p)^{m-1} p + \frac{1-p}{1-(1-p)} \\
&= 1 + 2\frac{1-p}{p}\mathbb{E}[X] + \frac{1-p}{p} \\
&= 1 + \frac{2(1-p)}{p^2} + \frac{1-p}{p} \quad \text{(since } \mathbb{E}[X]=1/p\text{)} \\
&= 1 + \frac{2}{p^2} - \frac{2}{p} + \frac{1}{p} - 1 \\
&= \frac{2}{p^2} - \frac{1}{p}.
\end{aligned}$$

So we obtain $pT = \frac{2}{p^2} - \frac{1}{p}$. Therefore Var($X$) is given by

$$\begin{aligned}
Var(X) &= \mathbb{E}[X^2] - \{\mathbb{E}[X]\}^2 \\
&= pT - \frac{1}{p^2} \quad \text{(since } \mathbb{E}[X^2] = pT \text{ as in (8) and} \mathbb{E}[X] = 1/p) \\
&= \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} \\
&= \frac{1}{p^2} - \frac{1}{p} \\
&= \frac{1-p}{p^2}.
\end{aligned}$$

Alternative Method: This method is similar as the alternative method described above. Differentiating the same function f (defined in (5)) twice with respect to p we get

$$f''(p) = \sum_{n=1}^{\infty} n(n-1)(1-p)^{n-2}$$

On the other hand, differentiating (6) twice with respect to p we get

$$f''(p) = \frac{2}{p^3}.$$

Therefore we have

$$\sum_{n=1}^{\infty} n(n-1)(1-p)^{n-2} = \frac{2}{p^3}. \tag{9}$$

Now we notice that

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{n=1}^{\infty} n^2(1-p)^{n-1}p \\
&= \sum_{n=1}^{\infty} [n(n-1)+n](1-p)^{n-1}p \\
&= p(1-p)\left[\sum_{n=1}^{\infty} n(n-1)(1-p)^{n-2}\right] + \sum_{n=1}^{\infty} n(1-p)^{n-1}p \\
&= p(1-p)\frac{2}{p^3} + \mathbb{E}[X] \quad \text{(using (9))} \\
&= \frac{2(1-p)}{p^2} + \frac{1}{p} \\
&= \frac{2}{p^2} - \frac{1}{p}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
Var(X) &= \mathbb{E}[X^2] - \{\mathbb{E}[X]\}^2 \\
&= \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} \\
&= \frac{1}{p^2} - \frac{1}{p} \\
&= \frac{1-p}{p^2}.
\end{aligned}$$