March 18, 2009.

FINAL EXAM

NAME(print in CAPITAL letters, first name first): __________________________

NAME(sign): __________________________________________________________

ID#: _________________________________________________________________

Instructions: Each of the 6 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Calculators, books or notes are not allowed. Do not evaluate binomial symbols, powers, etc. to give the result as a decimal number.

Make sure that you have a total of 7 pages (including this one) with 6 problems.

1
2
3
4
5
6

TOTAL
1. Recall that a full deck of cards contains 13 cards of each of the four suits (♠, ♦, ♥, ♣). Select cards from the deck at random, one by one, without replacement.

(a) Compute the probability that the first four cards selected are all hearts (♥).

\[
\binom{13}{4} \div \binom{52}{4}
\]

(b) Compute the probability that all suits are represented among the first four cards selected.

\[
\frac{13^4}{52^4}
\]

(c) Compute the expected number of different suits among the first four cards selected.

\[
X = \#\text{of different suits} = I_♥ + I_♦ + I_♣ + I_♠
\]

\[
E(I_♥) = P(♥ \text{ represented}) = 1 - \frac{\binom{39}{4}}{\binom{52}{4}}
\]

\[
\therefore \text{EX} = 4 \left(1 - \frac{\binom{39}{4}}{\binom{52}{4}}\right)
\]

(d) Extra credit, do not attempt unless you have time. Compute the expected number of cards you have to select to get the first hearts card.

Label ♥ cards 1, 2, ..., 39 and let

\[
I_i = \begin{cases} 
1 & \text{if card } i \text{ selected before any } ♥ \text{ card} \\
0 & \text{otherwise}
\end{cases}
\]

\[
E[I_i] = \frac{1}{14}
\]

\[
N = 1 + I_1 + \cdots + I_{39} \quad \text{EN} = 1 + \frac{39}{14} = \frac{53}{14}
\]
2. Eleven Scandinavians: 2 Swedes, 4 Norwegians and 5 Finns, are seated on a row of 11 chairs at random.
(a) Compute the probability that all groups sit together (i.e., Swedes occupy adjacent seats, and so do Norwegians and Finns).

\[
\frac{2!4!5!3!}{11!}
\]

(b) Compute the probability that at least one of the groups sits together.

\[
P(A_S \cup A_N \cup A_F) = P(A_S) + P(A_N) + P(A_F) - P(A_S \cap A_N) - P(A_S \cap A_F) - P(A_N \cap A_F) + P(A_S \cap A_N \cap A_F)
\]

\[
= \frac{1}{11!} \left( 2!10! + 4!8! + 5!7! - 2!4!7! - 2!5!6! - 4!5!4! + 2!4!5!3! \right)
\]

(c) Compute the probability that the two Swedes have exactly one person sitting between them.

Swedes occupy two chairs if they are separated by one person, the possible sitting arrangements are: 1-3, 2-4, ..., 9-11, exactly 9 of them.

Answer: \[ \frac{9}{11} = \frac{9.2!}{10.11} = \frac{9}{55} \]

\[ \left( \text{Or: } \frac{9.2!9!}{11!} \right) \]
3. You have two fair coins. Toss the first coin three times, and let $X$ be the number of heads. Then toss the second coin $X$ times, that is, as many times as you got heads on the first one. Let $Y$ be the number of heads on the second coin. (For example, if $X = 0$, $Y$ is automatically 0; if $X = 2$, toss the second coin twice and count the number of heads to get $Y$.)

(a) Determine the joint probability mass function of $X$ and $Y$, that is, write down a formula for $P(X = i, Y = j)$ for all relevant $i$ and $j$.

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Q5: \[ P(X = i, Y = j) = \binom{3}{i} \left( \frac{1}{2} \right)^i \left( \frac{1}{2} \right)^{j-i} \quad 0 \leq j \leq i \leq 3 \]

E.g., \[ P(X = 2, Y = 1) = P(X = 2) \cdot P(Y = 1 \mid X = 2) = \left( \frac{3}{2} \right)^{\frac{1}{2^3}} \cdot \left( \frac{3}{2} \right)^{\frac{1}{2^2}} = \frac{3 \cdot 2}{2^5} \]

(b) Compute $P(X \geq 2 \mid Y = 1)$.

\[
= \frac{P(X = 2, Y = 1) + P(X = 3, Y = 1)}{P(X = 1, Y = 1) + P(X = 2, Y = 1) + P(X = 3, Y = 1)}
\]

\[
= \frac{12 + 3}{12 + 12 + 3} = \frac{15}{27} = \frac{5}{9}
\]
4. Assume that 2,000,000 single tourists visit Las Vegas every year. Assume also that each of these tourists independently gets married while drunk 1/1,000,000.
(a) Write down the exact probability that exactly 3 tourists will get married while drunk in 2009.
\[
\binom{n}{3} p^3 (1-p)^{n-3} \quad \text{where} \quad p = \frac{1}{1000000} \quad \text{and} \quad n = 2,000,000.
\]

(b) Compute the expected number of drunk marriages in next 10 years.
\[
np = 10 \cdot \frac{1}{1000000} = 0.01
\]

(c) Write down a relevant approximate expression for the probability in (a).

\[
\text{Poisson, with } \lambda = np = 2
\]
\[
\frac{\lambda^3}{3!} e^{-\lambda} \approx \frac{4}{3} e^{-2}
\]

(d) Write down an approximate expression for the probability that there will be no drunk marriages during at least one of the next 3 years.
\[
1 - p(\text{at least one d.m. in each of next 3 years})
\]
\[
= 1 - (1 - e^{-\lambda})^3
\]
\[
= 1 - (1 - e^{-2})^3
\]
5. Toss a fair coin twice. You \textit{win} $2 if at least one of the two tosses comes out heads, \textit{lose} $1 if no toss comes out heads, and win or lose nothing otherwise.

(a) What is the expected number of games you need to play to win once?

\[ P(\text{win}) = \frac{3}{4}, \quad \text{so} \quad E(\text{geometric}) = \frac{4}{3} \]

(b) Assume that you play this game 500 times. What is, approximately, the probability that you win at least $135?

\[
\begin{align*}
E(X) &= 2 \cdot \frac{3}{4} - 1 \cdot \frac{1}{4} = \frac{5}{4} \\
\text{Var}(X) &= 4 \cdot \left(\frac{3}{4}\right)^2 + 1 \cdot \left(\frac{1}{4}\right)^2 - \left(\frac{5}{4}\right)^2 = \frac{13}{4} - \frac{25}{16} = \frac{27}{16} \\
S_n &= X_1 + \ldots + X_n \\
P\left( S_n \geq 135 \right) &= P\left( \frac{S_n - n \cdot \frac{5}{4}}{\sqrt{n \cdot \frac{27}{16}}} \geq \frac{135 - n \cdot \frac{5}{4}}{\sqrt{n \cdot \frac{27}{16}}} \right) \\
&\approx P\left( Z \geq \Phi^{-1}\left( \frac{135 - n \cdot \frac{5}{4}}{\sqrt{n \cdot \frac{27}{16}}} \right) \right) < 0 \\
\text{where} \quad n &= 500, \quad n \cdot \frac{5}{4} &= 625, \quad \text{and} \quad \Phi^{-1}\left( \frac{490}{\sqrt{100 \cdot \frac{27}{16}}} \right) = 0.001 \\
&= \Phi\left( \frac{490}{\sqrt{100 \cdot \frac{27}{16}}} \right)
\end{align*}
\]

(c) Again, assume that you play this game 500 times. Compute (approximately) the amount of money \( x \), such that your winnings will exceed \( x \) with probability 0.5. Then do the same with probability 0.9.

\[
P(S_n \geq x) \approx P\left( Z \geq \frac{x - 625}{\sqrt{100 \cdot \frac{27}{16}}} \right)
\]

For 0.5: \( x = 625 \)

For 0.9: \( \frac{625 - x}{\sqrt{100 \cdot \frac{27}{16}}} = 1.28 \)

\[
x = 625 - 1.28 \sqrt{100 \cdot \frac{27}{16}}
\]
6. Two random variables $X$ and $Y$ are independent and have the same probability density function

$$g(x) = \begin{cases} \frac{1}{x+1}, & x \in [0, 1], \\ 0, & \text{otherwise}. \end{cases}$$

(a) Find the value of $c$. Here and in (b): use $\int_0^1 x^n \, dx = \frac{1}{n+1}$, for $n > -1$.

$$M = c \int_0^1 (1+x) \, dx = c \left(1 + \frac{1}{2} \right) = \frac{3c}{2} \quad \Rightarrow \quad c = \frac{2}{3}$$

(b) Find $\text{Var}(X+Y)$.

$$\text{Var}(X) + \text{Var}(Y) = 2 \text{Var}(X) \quad \Rightarrow \quad \text{Var}(X) = \frac{5}{9}$$

$$EX = \frac{2}{3} \int_0^1 x(1+x) \, dx = \frac{2}{3} \left(\frac{1}{2} + \frac{1}{3} \right) = \frac{5}{18}$$

$$EX^2 = \frac{1}{3} \int_0^1 x^2(1+x) \, dx = \frac{2}{3} \left(\frac{4}{3} + \frac{1}{4} \right) = \frac{7}{18}$$

Answer: $2 \left(\frac{7}{18} - \left(\frac{5}{9}\right)^2\right) = \frac{13}{81}$

(c) Find $P(X+Y < 1)$ and $P(X+Y \leq 1)$. Here and in (d): when you get to a single integral involving powers, stop.

These probabilities are the same and equal

$$\int_0^1 dx \int_0^{1-x} \left(\frac{2}{3}\right)^2 (1+x)(1+y) \, dy$$

$$= \left(\frac{2}{3}\right)^2 \int_0^1 dx \ (1+x) \left[ (1-x) + \frac{(1-x)^2}{2} \right]$$

(d) Find $E|X-Y|.$

$$\left(\frac{2}{3}\right)^2 \int_0^1 dx \int_0^x (1+y)(1-x)(1+y) \, dy$$

$$= \left(\frac{2}{3}\right)^2 \cdot 2 \int_0^1 dx \int_0^x dy \ (x-y)(1+x)(1+y)$$

$$= 2 \left(\frac{2}{3}\right)^2 \int_0^1 \left[ (1+x)(x + \frac{x^2}{2}) - (1+x) \left(\frac{x^2}{2} + \frac{x^3}{3}\right) \right] dx$$