

Homework 3

1. There are three bags: A (contains 2 white and 4 red balls), B (8 white, 4 red) and C (1 white 3 red). You select one ball at random from each bag, observe that exactly two are white, but forget which ball came from which bag. What is the probability that you selected a white ball from bag A?
2. Pick three cards one by one at random from a full deck of 52 cards (a) without replacement and (b) with replacement. Compute the probability that the first card is a hearts ($\heartsuit$) card, the second card is an Ace, and the third card is a hearts card.
3. A bag initially contains 5 white and seven black balls. Each time you select a ball at random and return it to the bag together with two balls of the same color. Compute the probability that (a) first two selected balls are black and the next two are white and (b) that among the first four selected balls, exactly two are black.
4. There are two bags: A (contains 2 white and 4 red balls), B (1 white, 1 red). Select a ball at random from A, then put it into B. Then select a final ball at random from B. Compute (a) the probability that the final ball is white and (b) the probability that the transferred ball is white given that the final ball is white.
5. Shuffle a full deck of cards and turn them over one by one until an Ace appears. It turns out that this happened exactly when the 20th card was turned over. What is the probability that the next card is (a) Ace of spades, (b) two of clubs?
6. Shuffle a full deck of cards and divide it into two stacks of 26 cards. A card is taken from the top of the first stack, and, after its value is observed, put into the second stack. The second stack is then reshuffled, a card is dealt from the top, and its value observed. What is the probability that the two values are the same?
7. A bag contains 8 black and 4 white balls. Three players, A, B, and C, select a ball at random in this order A, B, C, A, B, C, etc., until a white ball is selected, and the player who does that is the winner. Compute the winning probabilities for the three players if the balls are (a) replaced after each selection and (b) are never replaced.

You should also do the five Problems in Section 4 of the book.

Solutions

1. $P(\text{w from A} | 2 \text{ w selected}) = P(\text{w from A, 2 w selected}) / P(2 \text{ w selected}) = (\frac{2}{6} \cdot \frac{8}{12} \cdot \frac{3}{4} + \frac{2}{6} \cdot \frac{4}{12} \cdot \frac{1}{4}) / ((\frac{2}{6} \cdot \frac{8}{12} \cdot \frac{3}{4} + \frac{2}{6} \cdot \frac{4}{12} \cdot \frac{1}{4} + \frac{4}{6} \cdot \frac{8}{12} \cdot \frac{1}{4})$.
2. (a) $P(\heartsuit, A, \heartsuit) = P(A\heartsuit, A \text{ but not } \heartsuit, \heartsuit) + P(\heartsuit \text{ but not } A, A\heartsuit, \heartsuit) + P(\heartsuit \text{ but not } A, A \text{ but not } \heartsuit, \heartsuit) = \frac{1}{52} \cdot \frac{3}{51} \cdot \frac{12}{50} + \frac{12}{52} \cdot \frac{1}{51} \cdot \frac{11}{50} + \frac{12}{52} \cdot \frac{3}{51} \cdot \frac{12}{50}$. (b) $\frac{1}{4} \cdot \frac{1}{13} \cdot \frac{1}{4}$, by independence.
3. (a) $\frac{7}{12} \cdot \frac{9}{14} \cdot \frac{5}{16} \cdot \frac{7}{18}$; (b) every sequence of two black and two white balls among the first four has the same probability, hence the answer is $\binom{4}{2}$ times the answer in (a).
4. Let F_w be the event that the transferred ball is white and W the event that the final ball is white. Then (a) $P(W) = P(F_w)P(W|F_w) + P(F_w^c)P(W|F_w^c) = \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9}$, and as the two summands are equal (b) $P(F_w|W) = \frac{1}{2}$.
5. (a) Tricky way: 21st card must first be an Ace, then the Ace of spades, so $\frac{3}{32} \cdot \frac{1}{4}$. Less tricky way: outcomes are permutations of cards, and

$$\frac{P(20\text{th card an Ace, 21st Ace of spades, no Aces among first 19 cards})}{P(20\text{th card an Ace, no Aces among first 19 cards})} = \frac{\binom{31}{2} \cdot 3! \cdot 48!}{\binom{32}{3} \cdot 4! \cdot 48!}.$$

- (b) Tricky way: 21st card must not be an Ace, then a particular one from among 48 cards, so $\frac{29}{32} \cdot \frac{1}{48}$. Less tricky way:

$$\frac{\binom{31}{3} \cdot 4! \cdot 47!}{\binom{32}{3} \cdot 4! \cdot 48!}.$$

6. Let I be the event that the interchanged card is selected and B the event that the two values are the same. We have

$$P(B) = P(I)P(B|I) + P(I^c)P(B|I^c) = \frac{1}{27} \cdot 1 + \frac{26}{27} \cdot \frac{3}{51}.$$

7. Call a *round* three consecutive selections, starting with A. For (a),

$$P(\text{A wins}) = P(\text{A wins} | \text{game decided on 1st round}) = \frac{\frac{4}{12}}{\frac{4}{12} + \frac{8}{12} \cdot \frac{4}{12} + \frac{8}{12} \cdot \frac{8}{12} \cdot \frac{4}{12}}.$$

For (b),

$$\begin{aligned} P(\text{A wins}) &= P(\text{A wins on 1st round}) + P(\text{A wins on 2nd round}) + P(\text{A wins on 3rd round}) \\ &= \frac{4}{12} + \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{4}{9} + \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{3}{7} \cdot \frac{4}{6}. \end{aligned}$$

Similarly for the other two players.