Interlude: Practice Midterm 1

This practice exam covers the material from the first four chapters. Give yourself 50 minutes to solve the four problems, which you may assume have equal point score.

1. Ten fair dice are rolled. What is the probability that:
   (a) At least one 1 appears.
   (b) Each of the numbers 1, 2, 3 appears exactly twice, while 4 appears four times.
   (c) Each of the numbers 1, 2, 3 appears at least once.

2. Five married couples are seated at random around a round table.
   (a) Compute the probability that all couples sit together (i.e., every husband-wife pair occupies adjacent seats).
   (b) Compute the probability that at most one wife does not sit next to her husband.

3. Consider the following game. The player rolls a fair die. If he rolls 3 or less, he loses immediately. Otherwise he selects, at random, as many cards from the full deck as the number that came up on the die. The player wins if all four aces are among the selected cards.
   (a) Compute the winning probability for this game.
   (b) Smith only tells you that he recently played this game once, and won. What is the probability that he rolled a 6 on the die?

4. A chocolate egg either contains a toy or is empty. Assume that each egg contains a toy with probability \( p \in (0, 1) \), independently of other eggs. Each toy is, with equal probability, red, white, or blue (again, independently of other toys). You buy 5 eggs. Let \( E_1 \) be the event that you get at most 2 toys, and \( E_2 \) the event that you get you get at least one red and at least one white and at least one blue toy (so that you have complete collection).
   (a) Compute \( P(E_1) \). Why is this probability very easy to compute when \( p = 1/2 \)?
   (b) Compute \( P(E_2) \).
   (c) Are \( E_1 \) and \( E_2 \) independent? Explain.
Solutions to Practice Midterm 1

1. Ten fair dice are rolled. What is the probability that:

   (a) At least one 1 appears.

   **Solution:**

   
   
   
   \[ 1 - P(\text{no 1 appears}) = 1 - \left( \frac{5}{6} \right)^{10}. \]

   (b) Each of the numbers 1, 2, 3 appears exactly twice, while 4 appears four times.

   **Solution:**

   \[ \binom{10}{2} \binom{8}{2} \binom{6}{2} = \frac{10!}{2! \cdot 4! \cdot 6!}. \]

   (c) Each of the numbers 1, 2, 3 appears at least once.

   **Solution:**

   Let \( A_i \) be the event that the \( i \)th number does not appear. We know the following:

   \[ P(A_1) = P(A_2) = P(A_3) = \left( \frac{5}{6} \right)^{10}, \]

   \[ P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_2 \cap A_3) = \left( \frac{4}{6} \right)^{10}, \]

   \[ P(A_1 \cap A_2 \cap A_3) = \left( \frac{3}{6} \right)^{10}. \]
Then,
\[
P(1, 2, \text{ and } 3 \text{ all appears at least once})
\]
\[
= P((A_1 \cup A_2 \cup A_3)^c)
\]
\[
= 1 - P(A_1) - P(A_2) - P(A_3)
\]
\[
+ P(A_1 \cap A_2) + P(A_2 \cap A_3) + P(A_1 \cap A_3)
\]
\[
- P(A_1 \cap A_2 \cap A_3)
\]
\[
= 1 - 3 \cdot \left( \frac{5}{6} \right)^{10} + 3 \cdot \left( \frac{4}{6} \right)^{10} - \left( \frac{3}{6} \right)^{10}.
\]

2. Five married couples are seated at random around a round table.

(a) Compute the probability that all couples sit together (i.e., every husband-wife pair occupies adjacent seats).

Solution:

Let \( i \) be an integer in the set \( \{1, 2, 3, 4, 5\} \). Denote each husband and wife as \( h_i \) and \( w_i \), respectively.

i. Fix \( h_1 \) to one of the seats.

ii. There are 9! ways to order the remaining 9 people in the remaining 9 seats. This is our sample space.

iii. There are 2 ways to order \( w_1 \).

iv. Treat each couple as a block and the remaining 8 seats as 4 pairs (where each pair are adjacent). There are 4! ways to seat the remaining 4 couples into 4 pairs of seats.

v. There are \( 2^4 \) ways to order \( h_i \) and \( w_i \) within the pair of seats.

Therefore, our solution is
\[
\frac{2 \cdot 4! \cdot 2^4}{9!}.
\]

(b) Compute the probability that at most one wife does not sit next to her husband.

Solution:

Let \( A \) be the event that all wives sit next to their husbands, and let \( B \) be the event that exactly one wife does not sit next to her husband. We know that \( P(A) = \frac{2^5 \cdot 4!}{9!} \) from part (a). Moreover, \( B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \), where \( B_i \) is the event that \( w_i \)
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does not sit next to \( h_i \) and the remaining couples sit together. Then \( B_i \) are disjoint and their probabilities are all the same. So we need to determine \( P(B_1) \).

i. Sit \( h_1 \) on a specified seat.

ii. Sample space is 9! for ordering the remaining 9 people into the remaining 9 seats.

iii. Consider each of remaining 4 couples and \( w_1 \) as 5 blocks.

iv. As \( w_1 \) cannot be next to her husband, you have 3 positions for \( w_1 \) in the ordering of the 5 blocks.

v. There are 4! ways to order the remaining 4 couples.

vi. There are \( 2^4 \) ways to order the couples within their blocks.

Therefore,

\[
P(B_1) = 3 \cdot 4! \cdot 2^4 \frac{9!}{9!}.
\]

Our answer is then

\[
5 \cdot 3 \cdot 4! \cdot 2^4 \frac{9!}{9!} + 2^5 \cdot 4! \frac{9!}{9!}.
\]

3. Consider the following game. The player rolls a fair die. If he rolls 3 or less, he loses immediately. Otherwise he selects, at random, as many cards from the full deck as the number that came up on the die. The player wins if all four aces are among the selected cards.

(a) Compute the winning probability for this game.

Solution:

Let \( W \) be the event that you win. Let \( F_i \) be the event that you roll \( i \), where 

\[ i = 1, \ldots, 6; \quad P(F_i) = \frac{1}{6}. \]

Since we lose if we roll a 1, 2, or 3, \( P(W|F_1) = P(W|F_2) = P(W|F_3) = 0. \) Moreover,

\[
P(W|F_4) = \frac{1 \binom{5}{4}}{\binom{52}{4}},
\]

\[
P(W|F_5) = \frac{\binom{5}{4}}{\binom{52}{4}},
\]

\[
P(W|F_6) = \frac{\binom{6}{4}}{\binom{52}{4}}.
\]

Therefore,

\[
P(W) = \frac{1}{6} \cdot \frac{1}{\binom{52}{4}} \left( 1 + \binom{5}{4} + \binom{6}{4} \right).
\]
(b) Smith only tells you that he recently played this game once, and won. What is the probability that he rolled a 6 on the die?

Solution:

\[
P(F_6|W) = \frac{\frac{1}{6} \cdot \frac{1}{\binom{6}{1}} \cdot \binom{6}{1}}{P(W)}
\]

\[
= \frac{\binom{6}{1}}{1 + \binom{6}{1} + \binom{6}{1}}
\]

\[
= \frac{15}{21}
\]

\[
= \frac{5}{7}
\]

4. A chocolate egg either contains a toy or is empty. Assume that each egg contains a toy with probability \( p \in (0, 1) \), independently of other eggs. Each toy is, with equal probability, red, white, or blue (again, independently of other toys). You buy 5 eggs. Let \( E_1 \) be the event that you get at most 2 toys, and \( E_2 \) the event that you get you get at least one red and at least one white and at least one blue toy (so that you have complete collection).

(a) Compute \( P(E_1) \). Why is this probability very easy to compute when \( p = 1/2 \)?

Solution:

\[
P(E_1) = P(0 \text{ toys}) + P(1 \text{ toy}) + P(2 \text{ toys})
\]

\[
= (1 - p)^5 + 5p(1 - p)^4 + \binom{5}{2} p^2(1 - p)^3.
\]

When \( p = \frac{1}{2} \),

\[
P(\text{at most 2 toys}) = P(\text{at least 3 toys})
\]

\[
= P(\text{at most 2 eggs are empty})
\]

Therefore, the probability, \( P(E_1) = P(E_1^c) \) and \( P(E_1) = \frac{1}{2} \).
(b) Compute $P(E_2)$.

**Solution:**

Let $A_1$ be the event that red is missing, $A_2$ the event that white is missing, and $A_3$ the event that blue is missing.

\[
P(E_2) = P((A_1 \cup A_2 \cup A_3)^C)
\]
\[
= 1 - 3 \cdot \left(1 - \frac{p}{3}\right)^5 + 3 \cdot \left(1 - \frac{2p}{3}\right)^5 - (1 - p)^5.
\]

(c) Are $E_1$ and $E_2$ independent? Explain.

**Solution:**

No. $E_1 \cap E_2 = \emptyset$. 