Math 135A, Winter 2010.
March 20, 2010.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ________________________________

ID#: __________________________

Instructions: Each of the 6 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Calculators, books or notes are not allowed. Unless you are directed to do so, or it is required for further work, do **not** evaluate binomial symbols, powers, etc. to give the result as a decimal number.

Make sure that you have a total of 7 pages (including this one) with 6 problems.

1	
2	
3	
4	
5	
6	
TOTAL	

1. Bob tosses a fair coin 10 times. He wins if all tosses are Heads or if all tosses are Tails. Otherwise, Alice wins. Note that $2^9 = 512$, and assume that in every year they play the game once in each of 256 "play-days."

(a) Compute the probability that Bob wins in one instance of this game.

$$P(\text{Bob wins}) = \frac{2}{2^{10}} = \frac{1}{512}$$

8

(b) Compute, using the relevant approximation, the probability that Bob wins exactly twice in a year.

# of wins for Bob $\sim$ Binomial $\left(256, \frac{1}{512}\right) \approx$ Poisson $\left(\frac{1}{2}\right)$

$$P(2 \text{ wins}) \approx \frac{\left(\frac{1}{2}\right)^2}{2!} e^{-1/2} = \underline{\underline{\frac{1}{8} e^{-1/2}}}$$

8

binom 4

(c) Under the title *Alice losing her touch*, newspapers report that "although Bob did not win even once last year, he already won at least twice in the first half of this year." Approximately how likely is the statement in quotes to happen by chance before the games started?

$$P(\text{no wins last year}, \geq 2 \text{ wins this year})$$

$$= P(\text{—}''\text{—}) \cdot P(\text{—}''\text{—})$$

9

$$\approx e^{-1/2} \cdot \left(1 - e^{-1/4} - \frac{1}{4} e^{-1/4}\right) = \underline{\underline{e^{-1/2}\left(1 - \frac{5}{4} e^{-1/4}\right)}}$$

$(\approx 1.6\,\%)$

(No. of wins in $\frac{1}{2}$ of the year is $\approx$ Poisson $\left(\frac{1}{4}\right)$.)

2. Fifteen Scandinavians: 4 Swedes, 5 Norwegians and 6 Finns, are seated at random around a table.

(a) Compute the probability that the group of Swedes ends up sitting together.

$$\frac{4! \cdot 11!}{14!}$$ 8

(b) Compute the probability that Norwegians have exactly one non-Norwegian person between them (i.e., they don't sit together, but all five of them are in 6 adjacent seats).

Sit a fixed Norwegian. Then:

$$\frac{1}{14!}\Big[\underset{\uparrow}{10} \cdot \underset{\uparrow}{9!} \cdot \underbrace{(6! - 2\cdot 5!)}_{\uparrow}\Big]$$

Choose a non-Norwegian to sit between them.

Arrange remaining people in a row.

The 6 people sit together in an order which does not have the non-Norwegian at the end. 8

$$= \frac{1}{14} \cdot 10! \cdot 4 \cdot 5!$$

(c) Compute the expected number of Finns that end up siting next to at least one Finn.

N = # of Finns seated next to at least one Finn

$N = I_1 + \dots + I_6$, $I_i = I_{\{\text{Finn } i \text{ sits next to a Finn}\}}$

$$EI_i = 1 - \frac{9}{14} \cdot \frac{8}{13} = \frac{55}{91}$$ 9

$$EN = 6 \cdot \frac{55}{91}$$

3. A math class has three sections with 40, 35 and 30 students. A committee of six students is chosen (without replacement, of course) at random from these 105.

(a) Compute the probability that all chosen students are from the same section.

$$\frac{\binom{40}{6}+\binom{35}{6}+\binom{30}{6}}{\binom{105}{6}}$$

8

(b) Compute the probability that a section is represented with three chosen students, another one with two chosen students, and another one with a single chosen student.

$$\frac{1}{\binom{105}{6}}\Big[\binom{40}{3}\binom{35}{2}\binom{30}{1}+\binom{40}{3}\binom{35}{1}\binom{30}{2}+\binom{40}{2}\binom{35}{3}\binom{30}{1}$$
$$+\binom{40}{2}\binom{35}{1}\binom{30}{3}+\binom{40}{1}\binom{35}{3}\binom{30}{2}+\binom{40}{1}\binom{35}{2}\binom{30}{3}\Big]$$

8

(c) Compute the probability that all sections are represented.

$A_i :=$ {section i is missing}

$$1-P(A_1\cup A_2\cup A_3)$$
$$=1-P(A_1)-P(A_2)-P(A_3)+P(A_1\cap A_2)+P(A_1\cap A_3)+P(A_2\cap A_3)$$
$$-P(A_1\cap A_2\cap A_3)$$
$$=\boxed{1-\frac{1}{\binom{105}{6}}\Big[\binom{65}{6}+\binom{70}{6}+\binom{75}{6}\Big]+\frac{1}{\binom{105}{6}}\Big[\binom{40}{6}+\binom{35}{6}+\binom{30}{6}\Big]}$$

9

-0

4. Roll a fair die repeatedly.

(a) Compute the expected number of rolls needed to get the first 6.

Geometric$\left(\frac{1}{6}\right)$, so expected number is $\underline{\underline{6}}$. 6

(b) Compute the probability that the number on the first roll is strictly larger than each number on subsequent three rolls.

Condition on the 1st roll, and use Bayes' first formula:

$$\frac{1}{6}\left[\underset{\substack{\uparrow\\ P(\cdots|\text{1st roll 1})}}{0} + \underset{\substack{\uparrow\\ P(\cdots|\text{first roll 2})}}{\left(\frac{1}{6}\right)^3} + \left(\frac{2}{6}\right)^3 + \left(\frac{3}{6}\right)^3 + \left(\frac{4}{6}\right)^3 + \underset{\substack{\uparrow\\ P(\cdots|\text{first roll 6})}}{\left(\frac{5}{6}\right)^3}\right]$$

... 6

(c) Compute the expected number of rolls among first 100 on which the number obtained is strictly larger than each number on subsequent three rolls.

By indicator trick:

$\underline{100 \cdot (\text{answer in (b)})}$ 6

(d) Determine the probability mass function of (X, Y), where X is the number of 6's rolled in the first 10 rolls, and Y is the number on the 10'th roll. (Write a formula for $P(X = i, Y = j)$ rather than a large table.)

$$P(X=i, Y=j) = \begin{cases} \binom{9}{i}\left(\frac{1}{6}\right)^{i+1}\left(\frac{5}{6}\right)^{9-i}, & j \neq 6, \\ \binom{9}{i-1}\left(\frac{1}{6}\right)^{i}\left(\frac{5}{6}\right)^{10-i}, & j = 6. \end{cases}$$

7

$i = 0, \ldots, 10$

$j = 1, \ldots, 6$

(Here, $\binom{9}{10} = 0$.)

5. For a year, a casino offers the following free game to all of its "guests." A bag contains 4 cards, with respective dollar amounts 0, 2, 4, and 8 written on them. Guests enter the casino one by one, and each independently grabs at random one of the cards from the bag and gets as many dollars as the number on the card.

(a) Alice, Bob, Carl, and Diane are first four guests. Compute the variance of their *combined* winnings.

$X =$ winnings in one game

$$EX = \frac{1}{4}(0+2+4+8) = \frac{7}{2}$$

$$\mathrm{Var}(X) = \frac{1}{4}(0+4+16+64) - \left(\frac{7}{2}\right)^2 = 21 - \frac{49}{4} = \frac{35}{4}$$

✓ 8

$$\mathrm{Var}(X_A + X_B + X_C + X_D) = 4\,\mathrm{Var}(X) = \underline{35}.$$

(b) Assume that 350,000 people will enter the casino during the year. Compute the expected amount of money m the casino will have to pay for this game.

$n = 350{,}000$

$S_n = X_1 + \cdots + X_n$ 8

$$m = ES_n = n \cdot \frac{7}{2} = 350{,}000 \cdot \frac{7}{2}.$$

(c) Keep the assumption from (b). Compute, approximately, the probability that the casino will have to pay more than $m + 3{,}500$ dollars. Then approximate the probability that the casino will have to pay more than $m - \frac{1}{4} \cdot 3{,}500$ dollars.

$$P(S_n \geq m + 3{,}500)$$

$$= P\left(\frac{S_n - n \cdot \frac{7}{2}}{\sqrt{n \cdot \frac{35}{4}}} \geq \frac{3{,}500}{\sqrt{n \cdot \frac{35}{4}}}\right)$$

$$\approx P\left(Z \geq \frac{3{,}500}{\frac{1}{2} \cdot 35 \cdot 100}\right) = P(Z \geq 2)$$ 9

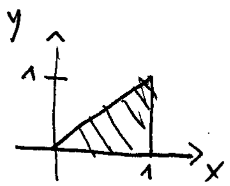
$$= 1 - \Phi(2) = 1 - 0.977 = \underline{0.023}$$

$$P(S_n \geq m - \frac{1}{4}\,3{,}500)$$

$$\approx P(Z \geq -0.5) = P(Z \leq 0.5)$$

$$= \Phi(0.5) \approx \underline{0.691}$$

6. A pair (X, Y) of random variables has the joint density



$$f(x,y) = \begin{cases} c(1+x), & 0 \le y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of c. Throughout, use that $\int_0^1 x^n \, dx = \frac{1}{n+1}$, for $n > -1$.

$$1 = c \int_0^1 dx \int_0^x (1+x)\, dy = c \int_0^1 (1+x)x \, dx = c\left(\tfrac{1}{2} + \tfrac{1}{3}\right) = c \cdot \frac{5}{6}$$ 6

$$\boxed{c = \frac{6}{5}}$$ $\frac{2}{3}(3)$

(b) Compute the marginal densities of X and Y. Are X and Y independent?

$$f_X(x) = \frac{6}{5} \int_0^x (1+x)\, dy = \underline{\frac{6}{5} x (1+x)}, \quad x \in [0,1] \ (0 \text{ otherwise})$$

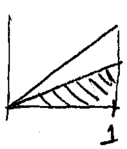
$$f_Y(y) = \frac{6}{5} \int_y^1 (1+x)\, dx = \frac{6}{5}\left(1 - y + \frac{1-y^2}{2}\right)$$ 7

$$= \frac{3}{5}\left(2 - 2y + 1 - y^2\right)$$

$$= \underline{\frac{3}{5}\left(3 - 2y - y^2\right)} \quad y \in [0,1] \ (0 \text{ otherwise})$$

$f(x,y) \ne f_X(x) f_Y(y)$ so <u>not</u> indep.

(c) Compute the conditional probability $P(4Y - X \le 0 | 2Y - X \le 0)$.



$$= \frac{P(Y \le \frac{1}{4}X)}{P(Y \le \frac{1}{2}X)} = \frac{\cancel{\frac{6}{5}} \int_0^1 dx \int_0^{x/4} (1+x)\, dy}{\cancel{\frac{6}{5}} \int_0^1 dx \int_0^{x/2} (1+x)\, dy} = \frac{\int_0^1 (1+x) \cdot \frac{x}{4}\, dx}{\int_0^1 (1+x) \cdot \frac{x}{2}\, dx}$$ 6

$$= \underline{\underline{\frac{1}{2}}}.$$

(d) Compute $E(XY)$.

$$\frac{6}{5} \cdot \int_0^1 dx \int_0^x xy(1+x)\, dy = \frac{6}{5} \int_0^1 x(1+x) \cdot \frac{x^2}{2}\, dx$$

$$= \frac{3}{5}\left(\frac{1}{4} + \frac{1}{5}\right) = \frac{3}{5} \cdot \frac{9}{20} = \underline{\underline{\frac{27}{100}}}$$ 6