Math 135A, Fall 2011.
Dec. 5, 2011.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 6 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a decimal number.

Make sure that you have a total of 7 pages (including this one) with 6 problems.

1	
2	
3	
4	
5	
6	
TOTAL	

1. (Recall: $\int_0^1 x^n\,dt = 1/(n+1)$ for $n > -1$.) The pair (X, Y) of random variables has joint distribution

$$f(x,y) = \begin{cases} cxy & \text{if } 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine c.

$$1 = c\int_0^1 dx \int_0^x xy\,dy = c\int_0^1 x\,dx \int_0^x y\,dy = c\int_0^1 \frac{x^3}{2}\,dx = \frac{c}{8}$$

$$\underline{\underline{c = 8}}$$

(b) Determine $E(XY)$.

$$E(XY) = 8\int_0^1 dx \int_0^x x^2y^2\,dy = 8\int_0^1 x^2 \cdot \frac{x^3}{3}\,dx = 8 \cdot \frac{1}{3 \cdot 6}$$

$$= \underline{\underline{\frac{4}{9}}}$$

(c) Determine the density of the random variable $Z = 1/X$.

$$P(Z<z) = P\left(\frac{1}{X} < z\right) = P\left(\frac{1}{z} < X\right)$$

$z > 1$

$$= 8\int_{1/z}^1 dx \int_0^x xy\,dy = 8\int_{1/z}^1 x \cdot \frac{x^2}{2}\,dx$$

$$= 4 \cdot \int_{1/z}^1 x^3\,dx$$

$$f_Z(z) = \frac{d}{dz}\left(4\int_{1/z}^1 x^3\,dx\right) = -4 \cdot \left(\frac{1}{z}\right)^3 \cdot -\frac{1}{z^2}$$

$$= \frac{4}{z^5} \quad \text{if } z > 1$$

(0 otherwise)

2. Every lotery ticket wins a prize with probability 1/10. Each day in November (30 days), Alice tosses a fair coin to decide whether she buys a lottery ticket. Let X be the number of days she buys the ticket and Y the number of prizes she wins.

(a) Determine the joint probability mass function of X and Y. (Write a formula rather than a table.)

$$P(X=x, Y=y) = \underbrace{\binom{30}{x} \frac{1}{2^{30}}}_{P(X=x)} \cdot \underbrace{\binom{x}{y} \cdot \left(\frac{1}{10}\right)^y \cdot \left(\frac{9}{10}\right)^{x-y}}_{P(Y=y \mid X=x)}$$

$0 \leq y \leq x \leq 30$

(b) Identify (as one of the well-known probability mass functions) the marginal probability mass function of X. Then do the same for Y. (*Hint.* Think rather than compute; no long computations are necessary.)

X is Binomial$(30, \frac{1}{2})$

Y is Binomial$(30, \frac{1}{20})$ (as every day she indep. wins the prize w.p. $\frac{1}{20}$)

(b) Compute the conditional probability mass function of X given $Y = 3$. Are X and Y independent?

$$P(X=x \mid Y=3) = \frac{\binom{30}{x} \frac{1}{2^{30}} \cdot \binom{x}{3} \left(\frac{1}{10}\right)^3 \left(\frac{9}{10}\right)^{x-3}}{\binom{30}{3} \left(\frac{1}{20}\right)^3 \left(\frac{19}{20}\right)^{27}}$$

$x \geq 3$

<u>Not</u> indep. The above prob. is 0 when $x=2$, but $P(X=2) \neq 0$.

3. A group of 20 Scandinavians consists of 5 Finns and 15 Danes. They are seated at random in a row of 20 chairs.

(a) What is the probability that all Finns sit together (i.e., that they occupy a row of five adjacent seats)?

$$\frac{5!\,16!}{20!}$$

(b) What is the probability that all 5 Finns sit on 8 leftmost chairs?

$$\frac{\binom{8}{5}}{\binom{20}{5}} \quad \text{or} \quad \frac{\binom{8}{5}\,5!\,15!}{20!}$$

(c) Assume that all Finns look to their right and all Danes look to their left. Let N be the number of Scandinavians that are observed by at least one neighbor. Compute EN.

$N = I_1 + \cdots + I_{20}$ $\qquad I_i = I_{\{i\text{th Sc. is observed by } \geq 1 \text{ nbr.}\}}$

$EI_1 = P(\text{D. on 2nd seat}) = \frac{3}{4}$

$EI_{20} = P(\text{F. on 19th seat}) = \frac{1}{4}$

$EI_i = 1 - P(\text{D. on seat } i-1,\ \text{F. on seat } i+1) = 1 - \frac{3}{4} \cdot \frac{5}{19}$

$1 < i < 20$

__Answer:__ $EN = \frac{3}{4} + \frac{1}{4} + 18\left(1 - \frac{3}{4} \cdot \frac{5}{19}\right)$

(d) Assume that all Finns still look to their right, but every Dane flips a fair coin to decide whether to look left or right. Let N be as in (c). Compute EN.

Now, $EI_1 = P(\text{D. on 2nd seat, looks left}) = \frac{3}{8}$

$EI_{20} = P(\text{F. on 19th seat}) + P(\text{D. on 19th seat, looks right})$

$= \frac{1}{4} + \frac{3}{8} = \frac{5}{8}$

$EI_i = 1 - P(\overleftarrow{D} \text{ on } i-1 \text{ \underline{and} } (F \text{ or } \overrightarrow{D} \text{ on } i+1))$

$1 < i < 20$

$= 1 - \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{5}{19} - \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{14}{19} \cdot \frac{1}{2}$

__Answer:__ $EN = \frac{3}{8} + \frac{5}{8} + 18\left(1 - \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{5}{19} - \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{14}{19} \cdot \frac{1}{2}\right)$

5. Each evening, Alice, Bob and Carol arrive at the casino, then proceed to play the following games (independently). Alice bets that she will toss four Heads in four coin tosses. Bob bets that he will roll two 6's in a roll of two dice. Carol bets that she will pick an Ace when she selects a card at random from a full deck of 52 cards .

(a) Compute the probability that tonight all three of them will win.

$$\frac{1}{16}\cdot\frac{1}{36}\cdot\frac{1}{13}$$

(b) Compute the probability that tonight at least ~~one~~ two of them will win.

$$P(\text{all win}) + P(A,B \text{ win; } C \text{ not}) + P(A,C \text{ win, } B \text{ not}) + P(B,C \text{ win, } A \text{ not})$$

$$= \frac{1}{16}\cdot\frac{1}{36}\cdot\frac{1}{13} + \frac{1}{16}\cdot\frac{1}{36}\cdot\frac{12}{13} + \frac{1}{16}\cdot\frac{35}{36}\cdot\frac{1}{13} + \frac{15}{16}\cdot\frac{1}{36}\cdot\frac{1}{13}$$

(c) Assume that they will do this for the next $1,872 = 4 \cdot 36 \cdot 13$ days. Estimate the probability that, on at least two of these days, all three will win.

no. of days with all three winners is $\text{Binomial}\left(1872, \frac{1}{16\cdot 36\cdot 13}\right) \approx \text{Poisson}\left(\frac{1}{4}\right)$

$$P(\text{no of such days} \geq 2) \approx 1 - e^{-1/4} - \frac{1}{4}e^{-1/4}$$

$$= 1 - \frac{5}{4}e^{-1/4}$$

4. In the *happiness game*, a player picks at random an integer N in the set $\{1,2,3,4\}$, then selects N cards from a full deck of 52 cards. Recall that a full deck contains 13 Hearts ($\heartsuit$) cards. The player is *happy* if at least one of the selected cards is a Hearts card.

(a) Bob plays this game once. Compute the probability that he is happy.

$$\frac{1}{4}\left[\frac{1}{4} + 1 - \frac{\binom{39}{2}}{\binom{52}{2}} + 1 - \frac{\binom{39}{3}}{\binom{52}{3}} + 1 - \frac{\binom{39}{4}}{\binom{52}{4}}\right]$$

$P(\text{happy} \mid N=1)$ $\qquad$ $P(\text{happy} \mid N=2)$ ...

(b) You see Bob after he played the game once and he is happy. What is the probability that the number N of selected cards was 4?

$$\frac{1 - \frac{\binom{39}{4}}{\binom{52}{4}}}{\frac{1}{4} + 1 - \frac{\binom{39}{2}}{\binom{52}{4}} + 1 - \frac{\binom{39}{3}}{\binom{52}{3}} + 1 - \frac{\binom{39}{4}}{\binom{52}{4}}}$$

(c) Now Bob plays this game until he is happy for the first time, then stops. What is the expected number of games he plays?

$$\frac{1}{\text{prob. in (a)}}$$

, as the number of games played is Geometric with success prob. given in (a).

6. A casino offers the following game. A player tosses two coins. If both tosses are Heads the player wins \$3 (i.e., the casino's revenue is -\$3), if both tosses are Tails the player wins \$1, otherwise the player loses \$4 (i.e., \$4 is the casino's revenue). Assume 3800 players play this game (independently). Let R be the resulting combined casino revenue.

(a) Compute ER and $\mathrm{Var}(R)$.

$$R = X_1 + \cdots + X_n \qquad n = 3800$$

X_i are i.i.d. with p.m.f $\quad P(X_i = -3) = \frac{1}{4}$

$P(X_i = -1) = \frac{1}{4}$

$EX_i = -3 \cdot \frac{1}{4} - 1 \cdot \frac{1}{4} + 4 \cdot \frac{1}{2} = \underline{\underline{1}} \qquad P(X_i = 4) = \frac{1}{2}$

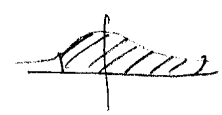
$$\mathrm{Var}(X_i) = 9 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 16 \cdot \frac{1}{2} - 1 = \frac{21}{2} - 1 = \underline{\underline{\frac{19}{2}}}$$

$\underline{ER = 3800}$, $\underline{\mathrm{Var}(R) = 3800 \cdot \frac{19}{2}}$

(b) Approximate the probability that R is at least \$3705. (Note: $95 = 5 \cdot 19$.)

$$P(R \geq 3705) \simeq P\left(\frac{R - ER}{\sqrt{\mathrm{Var} R}} \geq \frac{3705 - 3800}{\sqrt{3800 \cdot \frac{19}{2}}}\right)$$

$$\approx P\left(Z \geq \frac{-95}{\sqrt{19 \cdot 10^2 \cdot 19}}\right)$$



$$= P(Z \geq -0.5)$$

$$= P(Z \leq 0.5) \approx \underline{\underline{0.691}}$$