Math 135A, Fall 2013.
Dec. 10, 2013.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____________________

ID#: ________________

Instructions: Each of the 6 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 8 pages (including this one) with 6 problems.

1	
2	
3	
4	
5	
6	
TOTAL	

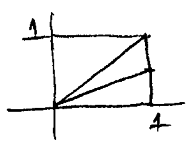
1. The pair (X, Y) of random variables has joint density given by $\cancel{f(x)}\ f(x,y) = \begin{cases} c(xy+1) & \text{if } x, y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

(a) Determine c. (Recall $\int_0^1 x^n\,dx = 1/(n+1)$ for $n > -1$.)

$$c \int_0^1 dx \int_0^1 (xy+1)\,dy = c \int_0^1 dx\,\left(x \cdot \tfrac{1}{2} + 1\right)$$

$$= c\left(\tfrac{1}{2}\cdot\tfrac{1}{2}+1\right) = c \cdot \tfrac{5}{4} \qquad \underline{\underline{c = \tfrac{4}{5}}}$$

(b) Determine $P(2Y \le X | Y \le X)$. $= \dfrac{P(Y \le \frac{X}{2})}{P(Y \le X)}$



$$= \frac{\cancel{\tfrac{4}{5}} \int_0^1 dx \int_0^{x/2} (xy+1)\,dy}{\cancel{\tfrac{4}{5}} \int_0^1 dx \int_0^{x} (xy+1)\,dy} = \frac{\int_0^1 dx\,\left(x \cdot \frac{x^2}{8} + \frac{x}{2}\right)}{\int_0^1 dx\,\left(x \cdot \frac{x^2}{2} + x\right)}$$

$$= \frac{\frac{1}{8}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}} = \frac{\frac{1}{16}+\frac{1}{2}}{\frac{1}{4}+1} = \underline{\underline{\frac{9}{20}}}$$

(c) Determine the density of the random variable $Z = \sqrt{X}$.

$$P(Z \le z) = P(X \le z^2) = \frac{4}{5}\int_0^{z^2} dx \int_0^1 (xy+1)\,dy$$

$z \in [0,1]$

$$= \frac{4}{5}\int_0^{z^2} \left(\tfrac{1}{2}x+1\right) dx$$

$$f_Z(z) = \frac{4}{5}\left(\tfrac{1}{2}z^2+1\right)\cdot 2z = \frac{4}{5}\left(z^3+2z\right) \quad \text{if } z \in [0,1]$$

$$= 0 \quad \text{otherwise}$$

2. Shuffle a full deck of 52 cards.

(a) What is the probability that all Aces are together in the deck (i.e., the four Aces are four consecutive cards, in any order)?

$$\frac{4!\ 49!}{52!} \qquad \left(\text{or} \qquad \frac{49}{\binom{52}{4}} \right)$$

(b) What is the probability that all the Aces are together in the deck and so are Kings, Queens, and Jacks.

$$\frac{(4!)^4 \cdot 40!}{52!}$$

(c) What is the probability that all the Aces are together in the deck and so are all the hearts (♡) cards?

$\boxed{\quad \heartsuit \quad}\boxed{A\heartsuit}\boxed{A}$ or $\boxed{A}\boxed{A\heartsuit}\boxed{\quad \heartsuit \quad}$

$52 - 16 = 36$
other cards

$$\frac{2 \cdot 3!\ 12! \cdot 37!}{52!}$$

Problem 2, continued.

(d) What is the probability that the first four cards are all hearts?

$$\frac{1}{4} \cdot \frac{12}{51} \cdot \frac{11}{50} \qquad \left(\text{or} \quad \frac{\binom{13}{3}}{\binom{52}{3}} \right)$$

(e) What is the probability that the third card is of different suit than either of the first two cards? (For example, this event happens if the suits are, in order, ♡♡♢ or ♡♠♢, but not in the case ♡♠♡.)

P(1st two card same suit, 3rd different suit)

+ P(1st three cards diff. suits)

$$= \underbrace{4}_{\substack{\text{choose}\\ \text{suit}\\ \text{of 1st 2 cards}}} \cdot \frac{1}{4} \cdot \frac{12}{51} \cdot \frac{39}{50} + \underbrace{4 \cdot 3 \cdot 2}_{\substack{\text{suits of}\\ \text{1st 3 cards}}} \; \frac{1}{4} \cdot \frac{13}{51} \cdot \frac{13}{50}$$

3. Roll five red dice and five blue dice. All dice are fair.
(a) Let X be the number of 6's rolled on red dice and Y the total number of 6's. Identify the probability mass functions of X and Y. Compute $\mathrm{Var}(Y)$.

X is Binomial$(5, \frac{1}{6})$

Y is Binomial$(10, \frac{1}{6})$

$$\mathrm{Var}(Y) = 10 \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{50}{36}$$

(b) Let X and Y be as in (a). Determine the joint probability mass function of X and Y. (Give a formula rather than a table.)

$$P(X=x, Y=y) = \underbrace{\binom{5}{x}\binom{5}{y-x}}_{\text{positions of 6's}} \underbrace{5^{10-y}}_{\text{other nos.}} \cdot \frac{1}{6^{10}}$$

$x = 0, \dots, 5$

$y = 0, \dots, 5$

$x \leq y \leq x+5$

(c) Give the conditional probability $P(X = 1|Y = 1)$ as a single fraction. Compute also the unconditional probability $P(X = 1)$. Are X and Y independent?

$P(X=1 \mid Y=1) = \frac{1}{2}$ (the 6 needs to apper on a red die)

$$P(X=1) = 5 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^4 = \frac{5^5}{6^5}$$

No. The two are different.

4. A bag contains 10 cards, labeled 1, 2, ..., 10. Each day, starting tomorrow, Alice selects three cards from the bag *with* replacement. Alice calls the day *fine* if the three cards have equal value and *great* if the three cards are all 1's.

(a) Compute the probability that tomorrow will be a fine day and the probability that it will be a great day.

$$P(\text{fine}) = 10 \cdot \frac{1}{10^3} = \frac{1}{10^2}$$

$$P(\text{great}) = \frac{1}{10^3}$$

(b) Let N be the number of days up to (and including) the first fine day. Identify the probability mass function of N and compute EN.

N is Geometric$\left(\frac{1}{10^2}\right)$, so $EN = 100$.

(c) Compute the probability that Alice's first fine day is a great day.

$$P(\text{great} \mid \text{fine}) = \frac{1}{10}.$$

(d) Approximate the probability that there are exactly 3 great days among the next 1,500 days.

No of great days is approx. Poisson $\left(\frac{3}{2}\right)$, so;

Answer. $\dfrac{e^{-\frac{3}{2}} \cdot \left(\frac{3}{2}\right)^3}{3!} = \underline{\underline{\dfrac{9}{16} e^{-3/2}}}.$

(e) Approximate the probability that there are at least 2 great days among the next 1,000 days, but after that there are no great days for the following 500 days.

By the same logic as above;

$$(1 - e^{-1} - e^{-1}) \cdot e^{-1/2} = \underline{(1 - 2e^{-1}) \cdot e^{-1/2}}$$

5. A math class of 100 students consist of three sections, with 50, 30, and 20 students. A committee a 6 students is selected at random (and, of course, without replacement) from the 100 students.
(a) Compute the probability that every section is represented by two students.

$$\frac{\binom{50}{2}\binom{30}{2}\binom{20}{2}}{\binom{100}{6}}$$

(b) Compute the probability that a section is represented by 5 or more students.

$$\frac{1}{\binom{100}{6}}\left[\binom{50}{5}\binom{50}{1}+\binom{50}{6}+\binom{30}{5}\binom{70}{1}+\binom{30}{6}+\binom{20}{5}\binom{80}{1}+\binom{20}{6}\right]$$

(c) Compute the probability that every section is represented by at least one student.

$$A_i = \{\text{section } i \text{ is not represented}\}$$

$$1 - P(A_1 \cup A_2 \cup A_3)$$

$$= 1 - P(A_1) - P(A_2) - P(A_3) + P(A_1 \cap A_2) + P(A_1 \cap A_3) + P(A_2 \cap A_3)$$

$$(P(A_1 \cap A_2 \cap A_3) = 0)$$

$$= 1 - \frac{\binom{50}{6}}{\binom{100}{6}} - \frac{\binom{70}{6}}{\binom{100}{6}} - \frac{\binom{80}{6}}{\binom{100}{6}} + \frac{\binom{20}{6}}{\binom{100}{6}} + \frac{\binom{30}{6}}{\binom{100}{6}} + \frac{\binom{50}{6}}{\binom{100}{6}}$$

6. A casino offers the following promotional game to its next $n = 480$ "guests." A bag contains 2 red and 3 blue balls. Each guest selects 2 balls from the bag, *without* replacement. If both selected balls are red, the guest wins \$7; if exactly one of the selected balls is red, the guest wins \$1; and if none of the selected balls is red, the guest wins \$$-1$ (i.e., pays \$1 to the casino). The balls are then returned to the bag for the next guest. Let the dollar winnings of each guest (from first to last) be $X_1, X_2, \ldots, X_n$ and $W = X_1 + \cdots + X_n$ the combined dollar amount of winnings.
(a) Determine the probability mass function of X_1, and compute EX_1 and $\mathrm{Var}(X_1)$. Write the results as simple fractions.

$$P(X_1 = 7) = \frac{1}{\binom{5}{2}} = 0.1$$
$$P(X_1 = 1) = \frac{6}{\binom{5}{2}} = 0.6$$
$$P(X_1 = -1) = \frac{3}{\binom{5}{2}} = 0.3$$

$$EX_1 = 0.7 + 0.6 - 0.3 = \underline{\underline{1}}$$

$$\mathrm{Var}(X_1) = 49 \cdot 0.1 + 1 \cdot 0.6 + 1 \cdot 0.3 - 1$$
$$= 4.9 + 0.9 - 1 = \underline{\underline{4.8}}$$

(b) Determine ~~as a simple fraction,~~ the expected number of guests, among the first $n-1$, who win strictly more money that the next guest.

Let $I_i = I_{\{i\text{th guest wins more than } (i+1)\text{'st}\}}$

then $EI_i = 0.1 \cdot 0.9 + 0.6 \cdot 0.3 = 0.27$

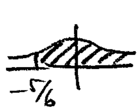
$$E(I_1 + \cdots + I_{n-1}) = (n-1) \cdot 0.27 = \underline{\underline{479 \cdot 0.27}}$$

(c) Using a relevant approximation, estimate the probability that W is at least \$440. Give the result as a decimal number.

$$P(W \geq 440) = P\left(\frac{W - 480}{\sqrt{480 \cdot 4.8}} > \frac{-40}{\sqrt{480 \cdot 4.8}}\right)$$

Z is $N(0,1)$

$$\approx P\left(Z \geq \frac{-40}{48}\right)$$



$$= P\left(Z \geq -\frac{5}{6}\right) = \Phi\left(\frac{5}{6}\right)$$
$$\approx \Phi(0.833) \approx \underline{\underline{0.797}}.$$