Math 135A, Winter 2013.
Mar. 23, 2013.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____________________

ID#: ____________________

Instructions: Each of the 6 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 9 pages (including this one) with 6 problems.

1	
2	
3	
4	
5	
6	
TOTAL	

1. The pair (X, Y) of random variables has density given by $f(x) = \begin{cases} c(x+y) & \text{if } x, y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

(a) Determine c. (Recall $\int_0^1 x^n\,dx = 1/(n+1)$ for $n > -1$.)

$$c \int_0^1 dx \int_0^1 dy\,(x+y) = c \int_0^1 \left(x + \tfrac{1}{2}\right) dx = c\left(\tfrac{1}{2} + \tfrac{1}{2}\right),$$

$$\underline{\underline{c = 1}}\,.$$

(b) Compute EX, EY, and $E(XY)$.

$$EX = EY = \int_0^1 dx \int_0^1 dy \cdot x(x+y)$$

$$= \int_0^1 x\left(x + \tfrac{1}{2}\right) dx = \tfrac{1}{3} + \tfrac{1}{2} \cdot \tfrac{1}{2} = \underline{\underline{\tfrac{7}{12}}}$$

$$E(XY) = \int_0^1 dx \int_0^1 xy(x+y)\,dy = \int_0^1 x\,dx \int_0^1 (xy + y^2)\,dy$$

$$= \int_0^1 x \cdot \left(x \cdot \tfrac{1}{2} + \tfrac{1}{3}\right) dx = \tfrac{1}{2} \cdot \tfrac{1}{3} + \tfrac{1}{3} \cdot \tfrac{1}{2} = \underline{\underline{\tfrac{1}{3}}}$$

(c) Determine the density of the random variable $Z = X^2$.

$$P(Z \le z) = P(X^2 \le z) = P(X \le \sqrt{z}) = \int_0^{\sqrt{z}} dx \int_0^1 (x+y)\,dy$$

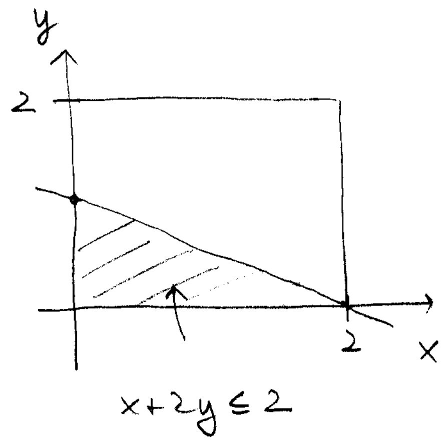
$$\underline{\underline{z \in (0,1)}}$$

$$= \int_0^{\sqrt{z}} \left(x + \tfrac{1}{2}\right) dx = \tfrac{1}{2}(x^2 + x) \Big|_0^{\sqrt{z}}$$

$$= \tfrac{1}{2}\left(z + \sqrt{z}\right)$$

$$\underline{\underline{f_Z(z) = \tfrac{1}{2} + \tfrac{1}{4\sqrt{z}}}}$$

2. (a) Choose a random point (X, Y) in the square $\{(x, y) : 0 \leq x, y \leq 2\}$ of side 2. Call this point *special* if $X + 2Y \leq 2$. Compute the probability that the so chosen point is special. (It is a good idea to draw a picture.)



Answer: $\underline{\underline{\frac{1}{4}}}$

(b) In the rest of this problem, assume that you choose 5 independent random points as in (a); for each special point roll two dice and for each other point roll one die. Let N be the number of dice rolled. Determine the probability mass function of N (write a formula rather than a table).

$P(N = n) = P(\,n-5$ special pts. out of 5$)$

$5 \leq n \leq 10$

$$= \binom{5}{n-5}\left(\frac{1}{4}\right)^{n-5} \cdot \left(\frac{3}{4}\right)^{5-(n-5)}$$

Problem 2, continued.

(c) Determine the probability that at least one 6 is rolled.

$$1 - P(\text{no } 6) = 1 - \sum_{n=5}^{10} P(N=n)\left(\frac{5}{6}\right)^n$$

or

Throw 5 dice and for each special pt. an additional die;

$$1 - P(\text{no } 6) = 1 - \left(\frac{5}{6}\right)^5 \cdot \left(\frac{3}{4} + \frac{1}{4} \cdot \frac{5}{6}\right)^5$$

not a special pt.

special but the roll is not 6

(d) Suppose you know that at least one 6 has been rolled. Determine the (conditional) probability that all 5 points were special.

$$\frac{P(\text{all } 5 \text{ pts. special } \& \text{ at least one } 6)}{P(\text{at least one } 6)}$$

$$= \frac{P(N=10)\left(1 - \left(\frac{5}{6}\right)^{10}\right)}{\text{Prob. in (c)}}$$

$$= \frac{\left(\frac{1}{4}\right)^5 \left(1 - \left(\frac{5}{6}\right)^{10}\right)}{\text{Prob. in (c)}}$$

3. Roll six fair dice.

(a) Compute the probability that you roll six different numbers.

$$\frac{6!}{6^6}$$

(b) Compute the probability that 4 is rolled exactly twice.

$$\frac{\binom{6}{4} 5^4}{6^6}$$

(c) Here and in (d), let N be the number of numbers that are rolled exactly twice. (E.g., if the rolls are 1, 1, 2, 4, 4, 6, then $N = 2$.) Compute EN.

$N = I_1 + \cdots + I_6$, where $I_i = I\{i \text{ is rolled exactly twice}\}$

$$EN = 6 \cdot (\text{prob. in (b)}) = 6 \cdot \frac{\binom{6}{4} 5^4}{6^6}$$

(d) Compute $P(N \geq 1)$ (i.e., the probability that at least one number is rolled exactly twice).

$$P(N \geq 1) = P\left(\bigcup_{i=1}^{6} A_i\right) \qquad A_i = \{i \text{ rolled exactly twice}\}$$

$$(\text{incl.-excl.}) = \underbrace{6 \cdot \frac{\binom{6}{4} 5^4}{6^6}}_{P(A_i)} - \binom{6}{2} \underbrace{\frac{\binom{6}{4}\binom{4}{2} 4^2}{6^6}}_{P(A_i \cap A_j)} + \binom{6}{3} \underbrace{\frac{\binom{6}{4}\binom{4}{2}}{6^6}}_{P(A_i \cap A_j \cap A_k)}$$

(Intersections of more than 3 A_i are empty.)

4. Five card players, among them Alice, are at the table. A full deck of 52 cards is shuffled and then each player is dealt two cards (with no replacement). Recall that there are 4 Aces among the 52 cards.
(a) Compute the probability that none of the players receives an Ace.

$$\frac{\binom{42}{4}}{\binom{52}{4}} \quad \text{or} \quad \frac{\binom{48}{10}}{\binom{52}{10}}$$

$\uparrow$ Position of A's

$\uparrow$ Choice of 10 cards that the 5 players get

(b) Compute the probability that four of the players receive one Ace each.

choice of the four players

$$\frac{\binom{5}{4} \cdot 2^4}{\binom{52}{4}}$$

$\leftarrow$ A 1st or 2nd card

$\uparrow$ no. of positions of A's

(c) Compute the probability that Alice receives two Aces.

$$\frac{4}{52} \cdot \frac{3}{51} = \frac{1}{13 \cdot 17}$$

Problem 4, continued.

(d) Compute the probability that at least one of the five players receives two Aces.

$A_i = \{\text{player } i \text{ receives two A's}\}$

$$P\left(\bigcup_{i=1}^{5} A_i\right) \underset{\substack{\uparrow \\ \text{incl.-excl.}}}{=} 5 \cdot \underbrace{\frac{1}{13 \cdot 17}}_{P(A_i)} - \binom{5}{2} \underbrace{\frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49}}_{P(A_i \cap A_j)}$$

(Intersections of more than 2 A_i's are empty.)

(e) Assume that the five players repeat this game $884 = 4 \cdot 13 \cdot 17$ times (independently) each month. Estimate the probability that, in the next month, Alice will receive two Aces at least twice.

No. of times Alice receives 2 A's is

$\text{Binomial}\left(884, \frac{1}{13 \cdot 17}\right) \approx \text{Poisson}(4)$

<u>Answer</u>: $1 - e^{-4} - 4e^{-4} = \underline{\underline{1 - 5e^{-4}}}$

5. A group of Scandinavians consists of 5 Swedes, 6 Danes, and 7 Finns. They are seated at random around a table.

(a) Compute the probability that all Swedes occupy adjacent chairs and the same is true of all Danes.

$$\frac{5!\ 6!\ 8!}{17!}$$

(b) Compute the expected number of Swedes who have $\underbrace{\text{a Dane as a neighbor and a Finn as a neighbor}}$,

$I_i = I_{\{i\text{th Swede has } \ldots\}}$

$$E(I_1+\cdots+I_5) = 5 \cdot \underset{\substack{\uparrow \\ \text{which is} \\ \text{on the left}}}{2} \cdot \underset{\substack{\uparrow \\ \text{choose} \\ \text{a Dane,}}}{\frac{6}{17}} \cdot \underset{\substack{\uparrow \\ \text{then a Finn}}}{\frac{7}{16}}$$

(c) Compute the probability that Finns sit in two contiguous groups (i.e., they are divided into two nonempty groups, each of each occupies adjacent chairs, but the two groups are not adjacent).

Sit a particular Swede first, then

sizes of two group

position the two groups so that they are not next to each other

$$\frac{6 \cdot \binom{11}{2}}{\binom{17}{6}}$$

choose chairs for Finns

(or replace $\binom{11}{2}$ by $\binom{12}{2} - 11$

all positions of two groups

positions in which two groups are neighbors

6. A casino offers the following promotional game to its next $n = 670$ "guests." A bag contains 3 red and 3 blue balls. without replacement. Each guest selects 3 balls from the bag without replacement. If all selected balls are blue, the guest has to pay \$10 to the casino (i.e., the guest wins \$−10). Otherwise, the guest wins as many dollars as the number of red balls selected. The balls are then returned to the bag for the next guest. Let the dollar winnings by each guest (from first to last) be $X_1, X_2, \ldots, X_n$ and $W = X_1 + \cdots + X_n$ the combined dollar amount of winnings.

(a) Determine the probability mass function of X_1, and compute EX_1 and $\mathrm{Var}(X_1)$.

$$P(X_1 = -10) = \frac{1}{\binom{6}{3}} = \frac{1}{20}$$

$$P(X_1 = 1) = \frac{\binom{3}{1}\binom{3}{2}}{\binom{6}{3}} = \frac{9}{20}$$

$$P(X_1 = 2) = \frac{9}{20}$$

$$P(X_1 = 3) = \frac{1}{20}$$

$$EX_1 = \frac{-10 + 9 + 18 + 3}{20} = \underline{\underline{1}}$$

$$EX_1^2 = \frac{100 + 9 + 36 + 9}{20} = \frac{154}{20} = 7.7$$

$$\mathrm{Var}(X_1) = \underline{\underline{6.7}}$$

(b) Determine, as a simple fraction, the probability that each of the first three guests wins at least \$2.

$$\left(\frac{9}{20} + \frac{1}{20}\right)^3 = \underline{\underline{\frac{1}{8}}}$$

(c) Using a relevant approximation, estimate the probability that W is at least \$700. Give the result as a decimal number. (Use $30/67 \approx 0.45$.)

$$P(W \geq 700) = P\left(\frac{W - 670}{\sqrt{670 \cdot 6.7}} \geq \frac{700 - 670}{\sqrt{670 \cdot 6.7}}\right)$$

$$\approx P\left(Z \geq \frac{30}{67}\right) \approx P(Z \geq 0.45)$$

$$= 1 - P(Z \leq 0.45)$$

$$\approx 1 - 0.67 = \underline{\underline{0.33}}$$