Math 135A, Fall 2017.
Dec. 14, 2017.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ________________

Instructions: Each of the 6 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 8 pages (including this one) with 6 problems.

1	
2	
3	
4	
5	
6	
TOTAL	

1. A pair (X, Y) of random variables has joint density $f(x,y) = \begin{cases} c(x+2y) & \text{if } x, y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

(a) Determine c. (Recall $\int_0^1 x^n \, dx = 1/(n+1)$ for $n > -1$.)

$$c \int_0^1 dx \int_0^1 dy \, (x+2y) = 1$$

$$c \int_0^1 (x+1) \, dx = 1$$

$$c\left(\frac{1}{2} + 1\right) = 1 \quad , \quad \underline{\underline{c = \frac{2}{3}}}$$

(b) Determine the marginal density of X.

$$f_X(x) = \frac{2}{3} \int_0^1 (x+2y)\,dy = \frac{2}{3}(x+1) \qquad 0 \le x \le 1$$

$$0 \qquad \text{otherwise}$$

(c) Compute $E(XY)$. Give the answer as a simple fraction.

$$E(XY) = \frac{2}{3} \int_0^1 x \, dx \int_0^1 y\,(x+2y)\,dy$$

$$= \frac{2}{3} \int_0^1 x \cdot \left(x \cdot \frac{1}{2} + \frac{2}{3}\right) dx$$

$$= \frac{2}{3} \cdot \left[\frac{1}{2} \cdot \frac{1}{3} + \frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{2}}\right] = \underline{\underline{\frac{1}{3}}}$$

2. Assume you have two bags: Bag 1 contains 3 red balls and 2 white balls, while Bag 2 contains 1 red ball and 5 white balls. At the first step, you select a ball at random from Bag 1. Afterwards, whenever you select a red ball, at the next step you select a ball at random from Bag 1, while whenever you select a white ball, at the next step you select a ball at random from Bag 2. (For example, if your second selected ball is white, regardless from which bag it came, you select a third ball from Bag 2.) There is never any replacement.

(a) Compute the probability that the first three selected balls are all white.

$$P(\underset{1}{w}\,\underset{2}{w}\,\underset{2}{w}) = \frac{2}{5}\cdot\frac{5}{6}\cdot\frac{4}{5} = \underline{\underline{\frac{4}{15}}}$$

(b) Compute the probability that the second selected ball is white.

$$P(\text{2nd } w) = P(\text{1st } w)\,P(\text{2nd } w \mid \text{1st } w) + P(\text{1st } r)\,P(\text{2nd } w \mid \text{1st } r)$$
$$= \frac{2}{5}\cdot\frac{5}{6} + \frac{3}{5}\cdot\frac{1}{2} = \frac{1}{3} + \frac{3}{10} = \underline{\underline{\frac{19}{30}}}$$

(c) Given that you know that the second selected ball is white, compute the (conditional) probability that the first selected ball is also white.

$$P(\text{1st } w \mid \text{2nd } w) = \frac{1/3}{19/30} = \underline{\underline{\frac{10}{19}}}$$

(d) Compute the probability that the third and fourth selected balls are both red.

RRRR impossible

$\underset{1\,2\,1\,1}{WRRR}$ $\quad \frac{2}{5}\cdot\frac{1}{6}\cdot\frac{3}{4}\cdot\frac{2}{3}$

$\underset{1\,1\,2\,1}{RWRR}$ $\quad \frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{6}\cdot\frac{2}{3}$

$\underset{1\,2\,2\,1}{WWRR}$ $\quad \frac{2}{5}\cdot\frac{5}{6}\cdot\frac{1}{5}\cdot\frac{3}{4}$

The answer is the sum of these three.

3. Roll a fair die 10 times. Let X be the largest number you roll in the *first 5* rolls, and Y the largest number you roll in *all 10* rolls.
(a) Write down the joint probability mass function of X and Y. (Write a formula rather than a table.)

$$P(X=i,\ Y=j) = \begin{cases} \dfrac{(i^5-(i-1)^5)\cdot i^5}{6^{10}} & i=j \\ \dfrac{(i^5-(i-1)^5)\cdot(j^5-(j-1)^5)}{6^{10}} & i<j \end{cases}$$

$$1 \le i \le j \le 6$$

(b) Compute $P(X=1)$ (best to do it directly, without using (a)) and the conditional probability $P(X=1|Y=2)$.

$$P(X=1) = \frac{1}{6^5}$$

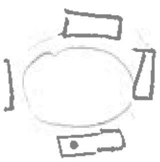
$$P(X=1, Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{\frac{2^5-1}{6^{10}}}{\frac{2^{10}-1}{6^{10}}} = \frac{2^5-1}{2^{10}-1}$$

(b) Are X and Y independent?

No. The above are not equal.

4. Every Saturday evening, four married couples (i.e, four husband-wife pairs) meet and sit at random in 8 chairs around a table.

(a) Call the evening *lucky* if every wife sits next to her husband. Compute the probability that next Saturday will be lucky. Write the result as a simple fraction. (Note: $3 \cdot 5 \cdot 7 = 105$.)



$$\frac{2^4 \cdot 3!}{7!} = \frac{2^4}{7 \cdot 6 \cdot 5 \cdot 4} = \underline{\underline{\frac{2}{105}}}$$

(b) Call the evening *almost lucky* if exactly three wives sit next to their husbands. Compute the probability that next Saturday will be almost lucky.

$$4\, P(\text{couples } 1,2,3 \text{ sit together, couple 4 not})$$
$$= 4\left[P(1,2,3 \text{ together}) - P(\text{all together})\right]$$
$$= 4 \cdot \left[\frac{2^3 \cdot 4!}{7!} - \frac{2}{105}\right] = 4\left[2 \cdot \frac{2}{105} - \frac{2}{105}\right]$$
$$= \underline{\underline{\frac{8}{105}}}$$

(c) For the next 105 Saturdays, what is the probability that 2 or more of them will be lucky? Use an appropriate approximation. $X = \#$ of lucky Sat,

$$\text{Binomial}\left(105, \frac{2}{105}\right) \approx \text{Poisson } (2)$$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$
$$\approx 1 - e^{-2} - 2e^{-2} = \underline{\underline{1 - 3e^{-2}}}$$

(d) Let N be the number of Saturdays up to (and including) the first almost lucky Saturday. Identify the probability mass function of N and compute EN.

$$N \text{ is Geometric}\left(\frac{8}{105}\right), \text{ so}$$

$$\underline{\underline{EN = \frac{105}{8}}}$$

5. A full deck of 52 cards is shuffled, then Alice, Bob, and Carol are each dealt 10 cards, without any replacement. (Full deck of 52 cards contains 13 cards of each of the four suits: ♡, ♢, ♠, ♣.)

(a) Compute the probability that all Alice's cards are of the same suit.

$$\underset{\substack{\uparrow \\ \text{Choose suit}}}{4} \cdot \frac{\binom{13}{10}}{\binom{52}{10}}$$

(b) Compute the probability that none of the three players gets a hearts (♡) card.

$$\frac{\binom{22}{13}}{\binom{52}{13}} \quad \leftarrow 52-30$$

↖ position of ♡

(c) Compute the probability that each of the three players gets at east one hearts (♡) card.

$E_A = \{A \text{ has no } \heartsuit\}$

$E_B = \{B \text{ — \" — }\}$

$E_C = \{C \text{ — \" — }\}$

$$P(E_A) = \frac{\binom{42}{13}}{\binom{52}{13}}$$

$$P(E_A \cap E_B) = \frac{\binom{32}{13}}{\binom{52}{13}}$$

$P(E_A \cap E_B \cap E_C) = $ in (b)

$$P((E_A \cup E_B \cup E_C)^c) = 1 - P(E_A \cup E_B \cup E_C)$$

$$= 1 - 3 \cdot \frac{\binom{42}{13}}{\binom{52}{13}} + 3\frac{\binom{32}{13}}{\binom{52}{13}} - \frac{\binom{22}{13}}{\binom{52}{13}}$$

Problem 5, continued

(d) Call a player *suited* if he/she has at least one card of each of the four suits. Compute the probability that Alice is suited.

$$E_{\heartsuit} = \{A. \text{ misses } \heartsuit\}$$

$E_{\diamondsuit}, E_{\spadesuit}, E_{\clubsuit}$ analogously

$$P(E_{\heartsuit}) = \frac{\binom{42}{13}}{\binom{52}{12}} = \frac{\binom{39}{10}}{\binom{52}{10}}$$

$$P(E_{\heartsuit} \cap E_{\diamondsuit}) = \frac{\binom{26}{10}}{\binom{52}{10}}$$

$$P(E_{\heartsuit} \cap E_{\diamondsuit} \cap E_{\spadesuit}) = \frac{\binom{13}{10}}{\binom{52}{10}}$$

$$P(E_{\heartsuit} \cap E_{\diamondsuit} \cap E_{\spadesuit} \cap E_{\clubsuit}) = 0,$$

$$P((E_{\heartsuit} \cup E_{\diamondsuit} \cup E_{\spadesuit} \cup E_{\clubsuit})^c) = 1 - 4\frac{\binom{39}{10}}{\binom{52}{10}} + 6\frac{\binom{26}{10}}{\binom{52}{10}} - 4\frac{\binom{13}{10}}{\binom{52}{10}}$$

(e) Compute the expected $\underbrace{\text{number of suited players}}_{X=}$.

By indicator trick, $EX = EI_{\{A.\text{ suited}\}} + EI_{\{B.\text{ suited}\}} + EI_{\{C.\text{ suited}\}}$

$= 3 \cdot (\text{answer in (d)})$

6. For years, Alice plays the following game once every day at the casino. She tosses two red and two blue coins. (All coin tosses are independent, as usual.) Call the *red count* the number of Heads tossed by red coins, and the *blue count* the number of Heads tossed by blue coins. If the red count is different from the blue count, she wins $-\$1$ (that is, she pays \$1 to the casino). Otherwise: if both counts are 2, she gets paid \$2; if both counts are 1, she gets paid \$1; and if both counts are 0 she wins or loses nothing. (For example, is she tosses 2 blue Heads and no red Heads, she loses \$1, but if she tosses Heads on all four coins, she wins \$2.) Let Alice's dollar winnings each day be $X_1, X_2, \ldots, X_n$ and $S_n = X_1 + \cdots + X_n$ the combined dollar amount of winnings after n days.

(a) Determine the probability mass function of X_1, and compute EX_1 and $\mathrm{Var}(X_1)$. Write the results as simple fractions.

x	-1	0	1	2
$P(X_1=x)$	$5/8$	$1/16$	$1/4$	$1/16$

$$EX_1 = -\frac{5}{8}+\frac{1}{4}+\frac{1}{8} = -\frac{1}{4}$$

$$\mathrm{Var}(X_1) = \frac{5}{8}+\frac{1}{4}+4\cdot\frac{1}{16}-\frac{1}{16} = \frac{17}{16}$$

(b) Compute $\mathrm{Var}(X_1 + X_2 + X_3 + X_4)$.

By independence: $4\,\mathrm{Var}(X_1) = \frac{17}{4}$

(c) Using a relevant approximation, estimate the probability that $S_{1700} \geq -408$, i.e., the probability that Alice's combined dollar losses in the first 1700 days are no more than $408 = 1700/4 - 17$. Give the result as a decimal number.

$$P(S_{1700} \geq -408) = P\left(\frac{S_{1700}+\frac{1}{4}\cdot 1700}{\sqrt{1700\cdot\frac{17}{16}}} \geq \frac{-408+\frac{1}{4}1700}{\sqrt{1700\cdot\frac{17}{16}}}\right)$$

by CLT:

$$\approx P\left(Z \geq \frac{17}{\frac{17\cdot 10}{4}}\right) = P(Z \geq 0.4)$$

$$= 1 - P(Z \leq 0.4)$$

$$\approx 1 - 0.66$$

$$= 0.33.$$