Math 135A, Winter 2010.
March 3, 2009.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____________________

ID#: ____________________

Instructions: Each of the 4 problems is worth 25 points. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Calculators, books or notes are not allowed. Unless specifically directed, do **not** evaluate binomial symbols, powers, etc. to give the result as a decimal number.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. Roll a fair die repeatedly. What is the probability that:
(a) In the first nine rolls, no 6 is rolled.

$$\left(\frac{5}{6}\right)^9$$

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(b) In the first nine rolls, a number appears twice, another number appears three times, and a third number appears four times.

$$\frac{6\cdot 5\cdot 4\binom{9}{2}\binom{7}{3}\binom{4}{4}}{6^9} = \frac{6\cdot 5\cdot 4\cdot \frac{9!}{2!\,3!\,4!}}{6^9}$$

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(c) In the first nine rolls, 6 appears exactly 3 times.

$$\binom{9}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^6 = \frac{\binom{9}{3}5^6}{6^9}$$

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(d) You get five different numbers on the first five rolls.

$$6 \cdot 5! \cdot \frac{1}{6^5} = \frac{5!}{6^4}$$

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choose a missing number

order the 5 present numbers

2. Shuffle a full deck of 52 cards.
(a) What is the probability that all four Aces end up together in the deck?

$$\frac{4!\cdot 49!}{52!} \quad \left(\text{or: } \quad \frac{49}{\binom{52}{4}} \right)$$

← choose leftmost position for the block of A's

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(b) What is the probability that all suits are represented in top 6 cards? Recall that each of the four suits (♡, ◇, ♣, ♠) has 13 representatives in the full deck.

$A_1 = \{\heartsuit \text{ missing}\}$

$A_2 = \{\diamondsuit \text{ -''- }\}$

$A_3 = \{\clubsuit \text{ -''- }\}$

$A_4 = \{\spadesuit \text{ -''- }\}$

$$P(A_1) = \frac{\binom{39}{6}}{\binom{52}{6}} = P(A_2) = P(A_3) = P(A$$

$$P(A_1 \cap A_2) = \frac{\binom{26}{6}}{\binom{52}{6}} = P(A_i \cap A_j),\ i<j$$

$$P(A_1 \cap A_2 \cap A_3) = \frac{\binom{13}{6}}{\binom{52}{6}} = P(A_i \cap A_j \cap A_k \quad i<j<k$$

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = 0$$

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By inclusion - exclusion:

$$P((A_1 \cup A_2 \cup A_3 \cup A_4)^c) = 1 - 4 \cdot P(A_1) + \binom{4}{2} P(A_1 \cap A_2) - \binom{4}{3} P(A_1 \cap A_2 \cap A_3)$$

$$= 1 - 4\frac{\binom{39}{6}}{\binom{52}{6}} + \binom{4}{2}\frac{\binom{26}{6}}{\binom{52}{6}} - \binom{4}{3}\frac{\binom{13}{6}}{\binom{52}{6}}$$

(c) What is the probability that the first card from the top is a Hearts (♡) card , the second is an Ace, and the third is a King?

$$P(\heartsuit, A, K) = P(A\heartsuit, A, K) + P(K\heartsuit, A, K) + P(\text{a } \heartsuit \text{ not A or K}, A, K)$$

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$$= \frac{1}{52}\cdot\frac{3}{51}\cdot\frac{4}{50} + \frac{1}{52}\cdot\frac{4}{51}\cdot\frac{3}{50} + \frac{11}{52}\cdot\frac{4}{51}\cdot\frac{4}{50}$$

3. A group of 30 Scandinavians includes 10 Swedes (among them 5 men and 5 women), 10 Norwegians (3 men and 7 women) and 10 Finns (6 men and 4 women). First choose a group at random (so that the three groups are equally likely), then choose a committee of 3 people from the chosen group at random.

(a) What is the probability that all members of the resulting committee are women?

$F_1 = \{\text{group S chosen}\}$ $\quad P(F_1) = P(F_2) = P(F_3) = \frac{1}{3}$

$F_2 = \{ \text{—''— N —''—} \}$

$F_3 = \{ \text{—''— F —''—} \}$

$A = \{\text{all women}\}$ $\quad P(A|F_1) = \dfrac{\binom{5}{3}}{\binom{10}{3}}$

$P(A|F_2) = \dfrac{\binom{7}{3}}{\binom{10}{3}}$ 13

$P(A|F_3) = \dfrac{\binom{4}{3}}{\binom{10}{3}}$

$$P(A) = \frac{1}{3} \cdot \frac{\binom{5}{3}}{\binom{10}{3}} + \frac{1}{3} \cdot \frac{\binom{7}{3}}{\binom{10}{3}} + \frac{1}{3} \frac{\binom{4}{3}}{\binom{10}{2}}$$

(b) The described procedure was performed in a closed room, and after it is over you are told that the committee consists of 3 women. What is now the probability that the committee consists of 3 Swedish women?

$$P(F_1|A) = \frac{\frac{1}{3} \frac{\binom{5}{3}}{\binom{10}{3}}}{\frac{1}{3} \frac{\binom{5}{3}}{\binom{10}{3}} + \frac{1}{3} \frac{\binom{7}{3}}{\binom{10}{3}} + \frac{1}{3} \frac{\binom{4}{3}}{\binom{10}{3}}}$$ 12

$$= \frac{\binom{5}{3}}{\binom{5}{3} + \binom{7}{3} + \binom{4}{3}}$$

4. Alice and Bob like to play the following game. Alice rolls three fair coins and Bob rolls a fair die. Alice wins if the number of Heads she tosses is strictly larger than the number on the die, and loses if her number of Heads is strictly smaller than Bob's number on the die. If the two numbers are equal, the game is repeated until it is decided.
(a) Compute the probability of the event A that Alice will win the game. Give the result as a simple fraction.

$D = \{\text{game decided on 1st round}\}$

$$P(A) = P(A \mid D) = \frac{P(A \cap D)}{P(D)} = \frac{5/48}{41/48} = \underline{\underline{\frac{5}{41}}}$$

$$P(D^c) = \underset{\text{Bob rolls 1}}{\frac{1}{6}} \cdot \underbrace{3 \cdot \frac{1}{8}}_{1H} + \underset{\text{Bob rolls 2}}{\frac{1}{6}} \cdot \underbrace{3 \cdot \frac{1}{8}}_{2H's} + \underset{\text{Bob rolls 3}}{\frac{1}{6}} \cdot \underbrace{\frac{1}{8}}_{3H's}$$ 9

$$= \frac{7}{48}, \quad \text{so } P(D) = \frac{41}{48}.$$

$$P(A \cap D) = \underset{\text{Bob rolls 1}}{\frac{1}{6}} \cdot \underbrace{(3+1) \cdot \frac{1}{8}}_{2 \text{ or } 3 H's} + \underset{\text{Bob rolls 2}}{\frac{1}{6}} \cdot \underbrace{\frac{1}{8}}_{3H's} = \frac{5}{48}$$

(b) Compute the probability of the event T that the game lasts at most two rounds (i.e., that Bob rolled his die either either once or twice).

$$P(T) = \frac{41}{48} + \frac{7}{48} \cdot \frac{41}{48}$$ 8

(c) Are A and T independent? Explain.

Yes. $P(A \mid T^c) = P(A)$, as the players face the same game if it isn't decided after 2 rounds. So A and T^c are independent, and so are A and T. 8