Math 135A, Winter 2011.
February 2, 2011.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Calculators, books or notes are not allowed. Unless you are directed to do so, or it is required for further work, do *not* evaluate complicated expressions to give the result as a decimal number.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. A fair die is rolled 10 times. What is the probability that:

(a) Neither 1 nor 2 is rolled on any of the rolls.

$$\left(\frac{4}{6}\right)^{10} = \left(\frac{2}{3}\right)^{10}$$

(6)

(b) Each of the numbers 1, 2, 3 appears exactly 3 times.

(6)

$$\frac{1}{6^{10}} \cdot \binom{10}{3}\binom{7}{3}\binom{4}{3} \cdot 3$$

rolls for 1,2,3

the 4th number

(c) Each of the numbers 1, 2, 3 appears at least 3 times.

$$\frac{1}{6^{10}} \cdot \left[\binom{10}{3}\binom{7}{3}\binom{4}{3} \cdot 3 + 3\binom{10}{4}\binom{6}{3}\binom{3}{3}\right]$$ (7)

each exactly 3

one of 1,2,3 appears 4 times, other 2 3 times

(c) Number 1 appears exactly four times, on four consecutive rolls.

(6)

$$\frac{1}{6^{10}} \cdot 7 \cdot 5^{6}$$

first roll

2. Four players, Alice, Bob, Carol, and Dan, are dealt 13 cards each from a shuffled deck of 52 cards. Recall that each suit (♡, ♢, ♣, ♠) has 13 distinct cards in the full deck.

(a) Compute the probability that Alice has no heart (♡) cards.

$$\frac{\binom{39}{13} \leftarrow \text{all ♡ go to other players}}{\binom{52}{13} \leftarrow \text{positions of ♡}}$$

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(b) Compute the probability that at least one of the four players has no heart (♡) cards.

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$$P(\text{A has no ♡}) = \frac{\binom{39}{13}}{\binom{52}{13}}$$

$$P(\text{A and B have no ♡}) = \frac{\binom{26}{13}}{\binom{52}{13}}$$

$$P(\text{A, B, C have no ♡}) = \frac{1}{\binom{52}{13}}$$

$$P(\text{at least one of A, B, C, D has no ♡})$$

$$= 4 \cdot \frac{\binom{39}{13}}{\binom{52}{13}} - \binom{4}{2} \frac{\binom{26}{13}}{\binom{52}{12}} + \binom{4}{3} \cdot \frac{1}{\binom{52}{13}}$$

3. A bag initially contains 3 white and 5 black balls. Each time you select a ball at random and return it to the bag together with a ball of the same color.
(a) Compute the probability that first two selected balls are black and the next two are white.

$$\frac{5}{8} \cdot \frac{6}{9} \cdot \frac{3}{10} \cdot \frac{4}{11}$$

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(b) Compute the probability that the second selected ball is white.

$F_W = \{1\text{st white}\}$ $\qquad S_W = \{2\text{nd white}\}$

$F_b = \{1\text{st black}\}$

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$$P(S_W) = P(S_W \mid F_W)\, P(F_W) + P(S_W \mid F_b)\, P(F_b)$$

$$= \frac{4}{9} \cdot \frac{3}{8} + \frac{3}{9} \cdot \frac{5}{8} = \frac{3}{8}$$

(c) You have just made your second selection and it is a white ball, but you have no memory of your first selection. What is the probability that your first selection was also white?

8

$$\frac{4 \cdot 3}{4 \cdot 3 + 3 \cdot 5} = \frac{4}{9}$$

4. Alice and Bob are married, and so are Carol and Dan. These four people, together with six other people, are seated at random on 10 chairs around a table. Let A be the event that Alice and Bob sit together; and C the event that Carol and Dan sit together.
(a) Compute the probability $P(A)$. Give the result as a simple fraction.

$P(A) = \frac{2}{9}$ (After you sit Alice, Bob must sit in 2 out of 9 remaining seats.) 6

(b) Compute the probability that both A and C happen, i.e., $P(A \cap C)$. Give the result as a simple fraction.

$$P(A \cap C) = \frac{2! \cdot 2! \cdot 7!}{9!} = \frac{4}{9 \cdot 8} = \frac{1}{18}$$

6

(c) Are A and C independent? Explain.

No. $P(A) = P(C) = \frac{2}{9}$, but 6

$P(A \cap C) = \frac{1}{18} \neq \left(\frac{2}{9}\right)^2 = P(A) \cdot P(C)$.

(d) Assume that the 10 people meet once every week, and at each meeting sit around the table at random (independently of the previous meetings). Alice will win a prize if, the first time Alice and Bob sit together, Carol and Dan do not sit together. Compute the probability that Alice wins the prize.

$$P(\text{win}) = P(\text{win} \mid \text{game decided})$$

$$= \frac{P(A \cap C^c)}{P(A)} = \frac{P(A) - P(A \cap C)}{P(A)}$$

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$$= \frac{2/9 - 1/18}{2/9} = \underline{\underline{\frac{3}{4}}}$$