Math 135A, Fall 2013.
Oct. 25. 2013.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________________

ID#: ____________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. Eight fair dice are rolled.

(a) Compute the probability that all numbers rolled are the same.

$$6/6^8 = \underline{\underline{\frac{1}{6^7}}}$$ 5

(b) Compute the probability that each of the four numbers 1, 2, 3, 4 is rolled exactly twice.

$$\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{6^8} = \underline{\underline{\frac{8!}{2^4 \cdot 6^8}}}$$ 5

(c) Compute the probability that exactly two numbers are represented among the numbers rolled (i.e., the set of numbers rolled contains exactly two elements).

choose the two numbers, which occur on each roll, but not only one of them in all rolls

$$\frac{1}{6^8} \cdot \binom{6}{2}\left(2^8 - 2\right)$$ 5

(d) Compute the probability that each of the four numbers 1, 2, 3, 4 is represented among the numbers rolled.

$A_i = \{\text{number } i \text{ missing}\}$

$P(A_1) = \left(\frac{5}{6}\right)^8$, $P(A_1 \cap A_2) = \left(\frac{4}{6}\right)^8$ 10

$P(A_1 \cap A_2 \cap A_3) = \left(\frac{3}{6}\right)^8$, $P(A_1 \cap A_2 \cap A_3 \cap A_4) = \left(\frac{2}{6}\right)^8$

<u>Answer</u>. $1 - P(A_1 \cup A_2 \cup A_3 \cup A_4)$

$$= 1 - 4 \cdot \left(\frac{5}{6}\right)^8 + \binom{4}{2}\left(\frac{4}{6}\right)^8 - \binom{4}{3}\left(\frac{3}{6}\right)^8 + \left(\frac{2}{6}\right)^8.$$

2. Ten people sit at a table in a restaurant and order the following dishes: 3 people order Asparagus, 2 people order Beef, and 5 people order Chicken. When the 10 dishes come, they are distributed to the 10 people at random.

(a) Compute the probability that every person received the correct order.

$$\frac{3!\,2!\,5!}{10!}$$

8

(b) Compute the probability that no person received the correct order.

The 5 C's must go to 5 people who did not order C.

8

$$\frac{5!\,\binom{5}{2}\,3!\,2!}{10!}$$

choose who gets B.

(c) Assume you know that every person who ordered Asparagus received the correct order. Compute the (conditional) probability that every person received the correct order. Give the result as a simple fraction.

$A = \{$people who ordered A get A$\}$

$B = \{$every gets the correct dish$\}$

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$$P(B|A) = \frac{P(B)}{P(A)} = \frac{\frac{3!\,2!\,5!}{10!}}{\frac{3!\,7!}{10!}} \leftarrow \text{from (a)}$$

$$= \frac{2\cdot 5!}{7!} = \frac{2}{7\cdot 6} = \underline{\underline{\frac{1}{21}}}.$$

3. Shuffle a full deck of 52 cards.

(a) What is the probability that the top three cards are an Ace, a King and a Queen, in any order?

$$\frac{4^3}{\binom{52}{3}}$$

8

(b) What is the probability that the top three cards are of three different suits?

choose suits, then cards from chosen suits

$$\frac{\binom{4}{3} \cdot 13^3}{\binom{52}{3}}$$

8

(c) Toss ~~four~~ three fair coins, note the number N of Heads and examine top N cards. (For example, if your tosses are all Tails, you examine no cards at all; if you toss 2 Heads and 1 Tails, you examine top 2 cards.) Compute the probability that there is at least one Ace among the examined cards.

$F_i = \{\text{toss } i \ H\text{'s}\}$

9

$A = \{\text{at least one } A\}$

$$P(A) = \sum_{i=0}^{3} P(F_i)\, P(A \mid F_i)$$

$$= \frac{1}{2^3} \cdot 0 + \frac{\binom{3}{1}}{2^3} \cdot \frac{1}{13} + \frac{\binom{3}{2}}{2^3} \cdot \left(1 - \frac{48}{52} \cdot \frac{47}{51}\right)$$

$$+ \frac{1}{2^3}\left(1 - \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50}\right)$$

4. Alice and Bob play the following *Max* game. A deck of 12 cards consists of 6 red cards, numbered $1, \ldots, 6$ and 6 blue cards, also numbered $1, \ldots, 6$. In the first round, three cards are selected at random from the deck. If the selected cards are of the same color, the game is decided: Alice wins if the maximal value of the selected cards is at least 5, and Bob wins otherwise. If the three cards are not of the same color (and thus the winner is not determined), the three selected cards are returned to the deck and another round is played. Rounds are repeatedly played until the game is decided.
(a) Compute the probability that the *Max* game is decided on the first round. Give the result as a simple fraction.

$$\frac{2\binom{6}{3}}{\binom{12}{3}} = \frac{2 \cdot 6 \cdot 5 \cdot 4}{12 \cdot 11 \cdot 10} = \underline{\underline{\frac{2}{11}}}$$

6

(b) Compute the probability that Alice wins the *Max* game. Give the result as a simple fraction.

$$P(\text{Bob wins}) = P(\text{Bob wins} \mid \text{game decided on 1st round})$$
$$= \frac{2\binom{4}{3}}{2\binom{6}{3}} = \frac{4 \cdot 3 \cdot 2}{6 \cdot 5 \cdot 4} = \frac{1}{5}$$

<u>Answer</u>: $\frac{4}{5}$ 7

(c) Compute the probability that the *Max* game lasts *exactly* three rounds.

$$\frac{9}{11} \cdot \frac{9}{11} \cdot \frac{2}{11}$$

6

(d) Assume that Alice and Bob play the *Max* game once a day, starting tomorrow. What is the probability that Alice wins three times before Bob wins twice?

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$$P(\text{Alice wins at least 3 of the first 4 games})$$
$$= \binom{4}{3}\left(\frac{4}{5}\right)^3 \cdot \frac{1}{5} + \left(\frac{4}{5}\right)^4 = \underline{\underline{2 \cdot \left(\frac{4}{5}\right)^4}}.$$