Math 135A, Fall 2013. Oct. 25, 2013.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, first name first):
VAME(sign):
D#:
nstructions: Each of the 4 problems has equal worth. Read each question carefully and answer in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. Unless directed to do so, do not evaluate complicated expressions to give the results a fraction or a decimal number. The proctor has been directed not to answer any interpretation uestions. Make sure that you have a total of 5 pages (including this one) with 4 problems.
1
2
3

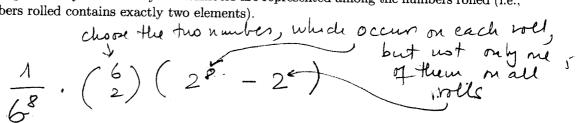
- 1. Eight fair dice are rolled.
- (a) Compute the probability that all numbers rolled are the same.

$$6/68 = \frac{1}{67}$$

(b) Compute the probability that each of the four numbers 1, 2, 3, 4 is rolled exactly twice.

$$\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{\binom{4}{2}} = \frac{2!}{2! \cdot 6!}$$

(c) Compute the probability that exactly two numbers are represented among the numbers rolled (i.e., the set of numbers rolled contains exactly two elements).



(d) Compute the probability that each of the four numbers 1, 2, 3, 4 is represented among the numbers rolled.

$$A_{i} = 4 \text{ number i missing } g$$

$$P(A_{1}) = \left(\frac{5}{6}\right)^{8} P(A_{1} \cap A_{2}) = \left(\frac{4}{6}\right)^{8} P(A_{1} \cap A_{2} \cap A_{3}) = \left(\frac{3}{6}\right)^{8} P(A_{1} \cap A_{2} \cap A_{3} \cap A_{4}) = \left(\frac{2}{6}\right)^{8}$$

Answer,
$$1 - P(A_1 \cup A_2 \cup A_3 \cup A_4)$$

= $4 - 4 \cdot (\frac{5}{6})^8 + (\frac{4}{2})(\frac{4}{6})^8 - (\frac{4}{3})(\frac{3}{6})^8 + (\frac{2}{6})^8$.

- 2. Ten people sit at a table in a restaurant and order the following dishes: 3 people order Asparagus, 2 people order Beef, and 5 people order Chicken. When the 10 dishes come, they are distributed to the 10 people at random.
- (a) Compute the probability that every person received the correct order.

(b) Compute the probability that no person received the correct order.

(c) Assume you know that every person who ordered Asparagus received the correct order. Compute the (conditional) probability that every person received the correct order. Give the result as a simple fraction.

A = Expense who ordered A get A?

B = Every gets the correct dish?

P(B|A) =
$$\frac{P(B)}{P(A)} = \frac{3!2!7!}{10!} < fum(a)$$

$$= \frac{2.5!}{7!} = \frac{2}{3.6} = \frac{1}{2!}$$

- 3. Shuffle a full deck of 52 cards.
- (a) What is the probability that the top three cards are an Ace, a King and a Queen, in any order?

$$\frac{4^{2}}{\binom{52}{3}}$$

(b) What is the probability that the top three cards are of three different suits?

three

(c) Toss fair coins, note the number N of Heads and examine top N cards. (For example, if your tosses are all Tails, you examine no cards at all; if you toss 2 Heads and 1 Tails, you examine top 2 cards.) Compute the probability that there is at least one Ace among the examined cards.

$$F_{i} = \{ \text{toss} \ i \ H's \}$$

$$A = \{ \text{at least one } A \}$$

$$P(A) = \sum_{i=0}^{3} P(F_{i}) P(A|F_{i})$$

$$= \frac{1}{2^{3}} \cdot 0 + \frac{\binom{3}{2}}{2^{3}} \cdot \frac{1}{13} + \frac{\binom{3}{2}}{2^{3}} \cdot \binom{1 - \frac{48}{52}}{51} \cdot \frac{47}{51}$$

$$+ \frac{1}{3^{3}} \left(1 - \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} \right)$$

- 4. Alice and Bob play the following Max game. A deck of 12 cards consists of 6 red cards, numbered 1,...,6 and 6 blue cards, also numbered 1,...,6. In the first round, three cards are selected at random from the deck. If the selected cards are of the same color, the game is decided: Alice wins if the maximal value of the selected cards is at least 5, and Bob wins otherwise. If the three cards are not of the same color (and thus the winner is not determined), the three selected cards are returned to the deck and another round is played. Rounds are repeatedly played until the game is decided.
- (a) Compute the probability that the Max game is decided on the first round. Give the result as a simple fraction.

$$\frac{2\binom{6}{3}}{\binom{12}{3}} = \frac{2.6.5.4}{12.11.10} = \frac{2}{11}$$

(b) Compute the probability that Alice wins the Max game. Give the result as a simple fraction.

$$P(Bob wins) = P(Bob wms | game decided m lst round)$$

$$= \frac{2(\frac{4}{3})}{2(\frac{6}{3})} = \frac{4 \cdot 3 \cdot 2}{6 \cdot 5 \cdot 4} = \frac{1}{5}$$
Anwer, $\frac{4}{5}$

(c) Compute the probability that the Max game lasts exactly three rounds.

(d) Assume that Alice and Bob play the Max game once a day, starting tomorrow. What is the probability that Alice wins three times before Bob wins twice?

P(Alice wins at least 3 of the first 4 games)
$$= \left(\frac{4}{3}\right)\left(\frac{4}{5}\right)^3 + \left(\frac{4}{5}\right)^4 = 2 \cdot \left(\frac{4}{5}\right)^4.$$