Math 135A, Winter 2013.
Feb. 6, 2013.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ________________________________

ID#: ________________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. Shuffle the full deck of 52 cards.

(a) Compute the probability that all 13 hearts ($\heartsuit$) cards are together in the deck (i.e., some consecutive 13 cards are hearts, in any order), and so are all the diamonds ($\diamondsuit$) cards.

$$\frac{(13!)^2\ 28!}{52!}$$

(b) Compute the probability that the first (i.e., the top) card in the deck is an Ace (of any suit), the second is a 2 (of any suit), and the third is a 3 (of any suit).

$$\frac{4^3 \cdot 49!}{52!} \quad \text{or} \quad \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50}$$

(c) Compute the probability that the first four cards are all hearts ($\heartsuit$).

$$\frac{\binom{13}{4}}{\binom{52}{4}} \quad \text{or} \quad \frac{\binom{48}{9}}{\binom{52}{13}} \quad \text{or} \quad \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49}$$

↑ set of firsts 4 cards

↑ position of 13 ♡

(d) As in poker, the values of cards are, in increasing order, 2, 3, ..., 10, J, Q, K, A. Compute the probability that the first card has strictly higher value than the second card. Give the result as a simple fraction. Let A be the event.

$$P(A) = \frac{1}{2} \cdot P(\text{1st two cards have different values})$$

$$= \frac{1}{2}\left(1 - \frac{3}{51}\right) = \frac{24}{51} = \underline{\underline{\frac{8}{17}}}$$

or

$$P(A) = P(\text{1st card } 2) \cdot P(A \mid \text{1st card } 2) + \cdots + P(\text{1st card } A)P(A \mid \text{1st card } A)$$

$$= \frac{1}{13}\left(0 + \frac{4\cdot 1}{51} + \frac{4\cdot 2}{51} + \cdots + \frac{4 \cdot 12}{51}\right) = \frac{4(1+\cdots+12)}{13 \cdot 51}$$

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$$= \frac{\overset{2}{\cancel{4}} \cdot 12 \cdot 13}{\cancel{2} \cdot 13 \cdot 51} = \frac{8}{17}$$

2. In the *Bag Experiment*, there are two bags: Bag 1 contains 3 red and 3 blue balls, and Bag 2 contains 2 red and 4 blue balls. All selections are at random without replacement. First, 3 balls are selected from Bag 1. The number of *blue* balls selected from Bag 1 is noted, and that many additional balls are then selected from Bag 2. (For example, if 1 red and 2 blue balls are selected from Bag 1, two balls from Bag 2 are added. If 3 red balls are selected from Bag 1, no balls from Bag 2 are added.) The Bag Experiment is *acceptable* if the total number of red balls selected (from both Bags) is 3.

Give all answers as ~~single~~ simple fractions.

(a) Compute the probability that the Bag Experiment is acceptable.

$F_i = \{i \text{ red balls from Bag 1}\}$

$A = \{\text{acceptable}\}$

$\binom{6}{3} = 20$

$\binom{6}{2} = 15$

i	$P(F_i)$	$P(A \mid F_i)$
0	$1/\binom{6}{3}$	0
1	$9/\binom{6}{3}$	$1/\binom{6}{2}$
2	$9/\binom{6}{3}$	$2/6$
3	$1/\binom{6}{3}$	1

$$P(A) = \frac{9}{20}\cdot\frac{1}{15} + \frac{9}{20}\cdot\frac{1}{3} + \frac{1}{20}$$

$$= \frac{9+45+15}{20\cdot 15} = \frac{69}{300} = \underline{\underline{\frac{23}{100}}}$$

(b) Assume that you only know the Bag Experiment was recently performed once and was acceptable. What is the (conditional) probability that two red balls were selected from Bag 1?

$$P(F_2 \mid A) = \frac{45}{69} = \underline{\underline{\frac{15}{23}}}$$

(c) Consider the following game. Perform the Bag Experiment. If it is acceptable, the game is decided: you win if the 3 selected red balls all came from Bag 1 and lose otherwise. If the experiment is not acceptable, you return all balls to their original bags and repeat. Compute the probability that you win.

$$P(\text{win} \mid \text{acceptable}) = P(F_3 \mid A) = \frac{15}{69} = \underline{\underline{\frac{5}{23}}}$$

3. Toss 10 independent coins. Let A be the event that you toss exactly 5 Heads.
(a) Compute $P(A)$ under the assumption that the coins are all fair.

$$\binom{10}{5} \cdot \frac{1}{2^{10}}$$

(b) Compute $P(A)$ under the assumption that 9 of the coins are fair, but one coin is unfair with Heads probability $1/3$.

$$P(A) = P(A \cap \{\text{unfair coin } H\}) + P(A \cap \{\text{unfair coin } T\})$$

$$= \frac{1}{3} \cdot \binom{9}{4} \frac{1}{2^9} + \frac{2}{3} \cdot \binom{9}{5} \frac{1}{2^9}$$

4. Assume a birthday is equally likely to be in any month of the year for any person (independently) in a group of 10 people.
(a) Compute the probability that all birthdays are in the same month.

$$\frac{12}{12^{10}} = \underline{\underline{\frac{1}{12^{11}}}}$$

(b) Compute the probability that January is represented 4 times, while February and March are represented 3 times each.

$$\frac{\binom{10}{4}\binom{6}{3}}{12^{10}} \quad \text{or} \quad \frac{\binom{10}{4\ 3\ 3}}{12^{10}}$$

(c) Compute the probability that at least one of the months is represented by exactly three birthdays.

$A_i = \{$ month i is represented exactly 3 times $\}$

$$P(A_i) = \frac{\binom{10}{3} 11^9}{12^{10}}$$

At most 3 of A_i can happen.

$$i<j: \ P(A_i \cap A_j) = \frac{\binom{10}{3}\binom{7}{3} 10^4}{12^{10}}$$

$$i<j<k: \ P(A_i \cap A_j \cap A_k) = \frac{\binom{10}{3}\binom{7}{3}\binom{4}{3} \cdot 9}{12^{10}}$$

Answer. $$P\left(\bigcup_{i=1}^{12} A_i\right) = 12 \cdot \frac{\binom{10}{3} 11^9}{12^{10}} - \binom{12}{2} \cdot \frac{\binom{10}{3}\binom{7}{3} 10^4}{12^{10}} + \binom{12}{3} \cdot \frac{\binom{10}{3}\binom{7}{3}\binom{4}{3} \cdot 9}{12^{10}}$$