Math 135A, Fall 2017.
Oct. 27, 2017.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. Roll a fair die repeatedly. (Note that in parts (a), (b), and (c), only the first 9 rolls matter.)
(a) Compute the probability that you do not roll a 6 on any of the first 9 rolls.

$$\left(\frac{5}{6}\right)^9$$

(b) Call a number $1, \ldots, 6$ a *fiver* if it occurs exactly 5 times among your first 9 rolls. Compute the probability that one of the numbers $1, \ldots, 6$ is a fiver.

$$\underset{\substack{\uparrow \\ \text{choose} \\ \text{a fiver}}}{6} \cdot \underset{\substack{\uparrow \\ \text{position} \\ \text{the fiver}}}{\binom{9}{5}} \cdot \underset{\substack{\uparrow \\ \text{fill other} \\ \text{positions}}}{5^4} \cdot \frac{1}{6^9}$$

(c) Call a number $1, \ldots, 6$ a *triple* if it occurs exactly 3 times among your first 9 rolls. Compute the probability that at least one of the numbers $1, \ldots, 6$ is a triple.

$A_i = \{i \text{ is a triple}\} \qquad i = 1, \ldots, 6$

$$P(A_i) = \frac{1}{6^9} \cdot \binom{9}{3} \cdot 5^6$$

$i < j$: $$P(A_i \cap A_j) = \frac{1}{6^9} \binom{9}{3}\binom{6}{3} 4^3$$

$i < j < k$: $$P(A_i \cap A_j \cap A_k) = \frac{1}{6^9} \binom{9}{3}\binom{6}{3}$$

By inclusion-exclusion:

$$P\left(\bigcup_{i=1}^{6} A_i\right) = \frac{1}{6^9}\left[6 \cdot \binom{9}{3} 5^6 - \binom{6}{2}\binom{9}{3}\binom{6}{3} 4^3 + \binom{6}{3}\binom{9}{3}\binom{6}{3}\right]$$

(d) Compute the probability that you roll 6 three times before you roll 5 twice.

Disregard rolls on which you do not roll 5 or 6. In the first 4 rolls on which you do roll 5 or 6, you need to roll ≥ 3 6's:

$$\left[\binom{4}{3} + \binom{4}{4}\right] \cdot \frac{1}{2^4} = \frac{5}{16}$$

2. Shuffle a full deck of 52 cards, and distribute them to four players, Alice, Bob, Carol, and Dan; therefore, each player gets 13 cards.

(a) Compute the probability that all of 13 Alice's cards are hearts (♡).

$$\frac{1}{\binom{52}{13}}$$

(b) Compute the probability that exactly 5 of 13 Alice's cards are hearts.

$$\frac{\binom{13}{5}\cdot\binom{39}{8}}{\binom{52}{13}}$$

(c) Compute the probability that one of the players has 4 hearts and the remaining three players have 3 hearts each.

$$4 \cdot \frac{\binom{13}{4}\binom{13}{3}^3}{\binom{52}{13}}$$

player with 4 ♡

positions of ♡

3. A bag initially contain 2 green and 5 red balls. Repeatedly perform the following steps. At each step, you select a ball at random from the bag, remove the selected ball from the bag, and then put a new *green* ball into the bag. For example, if your first selection happens to be a green ball, then the bag still contains 2 green and 5 red balls after the first step is completed; on the other hand, if your first selection is a red ball, then the bag contains 3 green and 4 red balls after the first step is completed.

(a) Compute the probability that the selected balls in first three steps are all red.

$$\frac{5}{7} \cdot \frac{4}{7} \cdot \frac{3}{7}$$

$A = \{ \quad \}$

(b) Compute the probability that the selected ball on the second step is red.

$G = \{$ green on 1st step $\}$
$R = \{$ red $-\|-$ $\}$

$P(G) = \frac{2}{7}$ $\quad P(A|G) = \frac{5}{7}$ $\quad P(A) = \frac{2}{7} \cdot \frac{5}{7} + \frac{5}{7} \cdot \frac{4}{7}$

$P(R) = \frac{5}{7}$ $\quad P(A|R) = \frac{4}{7}$ $\quad = \frac{30}{49}$

(c) Assuming the selected ball on the second step is red, what is the conditional probability that the selected ball on the first step is green? Give the result as a simple fraction.

Bayes : $\dfrac{2 \cdot 5}{2 \cdot 5 + 5 \cdot 4} = \underline{\underline{\dfrac{1}{3}}}$

4. Eight Scandinavians, 4 Finns, 3 Danes, and one Swede, play the following *Group* game. In each round, they are seated at random on 8 chairs around the table. The Finns automatically win if they sit together. If the Finns do not sit together, but Danes do sit together, then the Danes win. Otherwise, the game is undecided; all 8 of them get up and play another round. Rounds are played until the *Group* game is decided. (Note that the Swede never wins.)

(a) Compute the probability that the *Group* game is decided on the first round. Give the result as a simple fraction.

$$P(\text{Fs together}) = \frac{4!\,4!}{7!} = \frac{4}{35}$$

$$P(\text{Ds together}) = \frac{3!\,5!}{7!} = \frac{5}{35}$$

$$P(\text{Fs and Ds both together}) = \frac{4!\,3!\,2!}{7!} = \frac{2}{35}$$

$$P(\text{game decided on 1st round}) = \frac{4}{35} + \frac{5}{35} - \frac{2}{35} = \underline{\underline{\frac{1}{5}}}$$

(b) Compute the probability of the event A that the Finns win the *Group* game. Give the result as a simple fraction.

$$P(\text{Fs together} \mid \text{game decided}) = \frac{\frac{4}{35}}{\frac{1}{5}} = \underline{\underline{\frac{4}{7}}}$$

(c) Compute the probability of the event T that the *Group* game lasts *exactly* five rounds.

$$\left(\frac{4}{5}\right)^4 \cdot \frac{1}{5}$$

(d) Are A and T independent?

$\boxed{\text{Yes.}}$ The undecided rounds do not influence the prob. of win.