Math 135A, Winter 2010.
March 3, 2009.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 4 problems is worth 25 points. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Calculators, books or notes are not allowed. Unless specifically directed, do **not** evaluate binomial symbols, powers, etc. to give the result as a decimal number.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. The density function of a random variable X is given by

$$f(x) = \begin{cases} ax^2 + bx & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here a and b are constants. You also know that $EX = \frac{1}{2}$. (Below, you may use that $\int_0^1 x^n\,dx = \frac{1}{n+1}$ for $n \neq -1$.)

(a) Determine a and b.

$1 = \int_0^1 f(x)\,dx = \frac{1}{3}a + \frac{1}{2}b$ $\qquad$ $2a + 3b = 6$

$\frac{1}{2} = \int_0^1 x f(x)\,dx = \frac{1}{4}a + \frac{1}{3}b$ $\qquad$ $3a + 4b = 6$

$a + b = 0, \quad b = -a, \quad \boxed{\begin{array}{l} a = -6 \\ b = 6 \end{array}}$

$$f(x) = \begin{cases} 6(-x^2 + x), & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Determine $\mathrm{Var}(X)$.

$$E(X^2) = \int_0^1 x^2 f(x)\,dx = 6\left(-\frac{1}{5} + \frac{1}{4}\right) = 6 \cdot \frac{1}{20} = 0.3$$

$$\mathrm{Var}(X) = E(X^2) - (EX)^2 = 0.3 - 0.25 = \underline{\underline{0.05}}$$

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(c) Determine the density of the random variable $Y = 1/X$.

$\underline{y \in (1, \infty)}$: $P(Y < y) = P\left(\frac{1}{X} < y\right) = P\left(X > \frac{1}{y}\right) = \int_{1/y}^{1} 6(-x^2 + x)\,dx$

$$\underline{\underline{f_Y(y)}} = 6\left(-\frac{1}{y^2} + \frac{1}{y}\right) \cdot \frac{1}{y^2}$$

$$= \underline{\underline{6 \cdot \frac{y-1}{y^4} \quad \text{if} \quad y > 1}} \quad (\text{and } 0 \text{ otherwise})$$

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2. An urn contains 10,000 balls: 2,000 are red, 2 are white and the rest are blue. Select a ball from the urn 10,000 times *with replacement*. Let X be the number of times you select a red ball and Y the number of times you select a white ball.

(a) Estimate, using a relevant approximation, the probability that you get a white ball at least 3 times.

X is Binomial $\left(10{,}000, \frac{2}{10{,}000}\right) \approx$ Poisson (2).

$$P(X \geq 3) = 1 - P(X \leq 2) = 1 - P(X=0) - P(X=1) - P(X=2)$$ 12

$$\approx 1 - e^{-2} - 2e^{-2} - \frac{2^2}{2} e^{-2} = \underline{\underline{1 - 5e^{-2}}}.$$

(b) Estimate, using a relevant approximation, the probability that you get a red ball at least 2,080 times. (Use the tables to give the result as a decimal number.)

X is Binomial $\left(\overset{n}{10{,}000}, \overset{p}{\frac{1}{5}}\right)$. Use Normal approx.:

$$P(X \geq 2080) = P\left(\frac{X - 2000}{\sqrt{\ }40} \geq \frac{2080 - 2000}{40}\right)$$

$$\left(np = 2000, \ \sqrt{np(1-p)} = \sqrt{10000 \cdot \frac{1}{5} \cdot \frac{4}{5}} = 40.\right)$$ 13

$$\approx P(Z \geq 2) = 1 - \Phi(2)$$

$$\approx 1 - 0.9772 = 0.0228$$

3. Roll a die three times, and let X be the number rolled the first time and Y the maximum number on the three rolls.

(a) Determine the joint probability mass function of X and Y.

For $i, j = 1, 2, 3, 4, 5, 6$:

$$P(X=i,\ Y=j) = \begin{cases} \frac{1}{6} \cdot \left(\frac{i}{6}\right)^2 = \frac{i^2}{6^3} & \text{if } i = j, \\ \frac{1}{6} \cdot \left[\left(\frac{j}{6}\right)^2 - \left(\frac{j-1}{6}\right)^2\right] = \frac{2j-1}{6^3} & \text{if } i < j, \\ 0 & \text{otherwise}. \end{cases}$$

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(b) Determine $P(X+Y \leq 4)$.

$$= P(X=1, Y=1) + P(X=1, Y=2) + P(X=2, Y=2) + P(X=1, Y=3)$$

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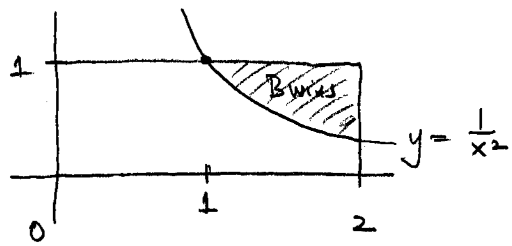
$$= \frac{1}{6^3} + \frac{3}{6^3} + \frac{4}{6^3} + \frac{5}{6^3} = \frac{13}{6^3} = \underline{\underline{\frac{13}{216}}}.$$

(c) Are X and Y independent? Explain.

No. $P(X=2, Y=1) = 0 \neq \underbrace{P(X=2)}_{=\frac{1}{6}} \cdot \underbrace{P(Y=1)}_{=\frac{1}{6^3}}$. 5

4. Alice and Bob like to play the following game. Alice chooses a random number X uniformly on $[0, 2]$, and Bob chooses, independently, a random number Y uniformly on $[0, 1]$. Alice wins if $1/X^2 > Y$.

(a) Compute the probability of the event that Alice wins. (Draw a picture, and give the result as a simple fraction.)



$P(B \text{ wins})$

$$= \frac{1}{2}\int_1^2 \left(1 - \frac{1}{x^2}\right) dx = \frac{1}{2}\left[1 + \frac{1}{x}\Big|_1^2\right]$$

$$= \frac{1}{4}$$

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So $P(A \text{ wins}) = \frac{3}{4}$!

(b) Compute the probability that, in the next 30 games, Alice will win exactly 5 times.

$$\binom{30}{5}\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)^{25}.$$

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(b) Alice just told you that they played once, and she won. What is now the probability that the number she chose was less than 1/2?

$$P\left(X \le \frac{1}{2} \,\middle|\, A \text{ wins}\right) = \frac{P\left(X \le \frac{1}{2}\right)}{P(A \text{ wins})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

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(If $X \le \frac{1}{2}$, A always wins.)