Math 135A, Fall 2011.
Nov. 18, 2011.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a decimal number.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. (Recall: $\int_0^1 t^n\,dt = 1/(n+1)$.) Alice and Bob will arrive at a bar on a Friday afternoon at *independent* random times after 5 pm. (We assume 5 pm to be time 0 from now on.)

Alice's arrival time T_1 has distribution (in hours) $f(t) = \begin{cases} c(t-t^2) & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$

Bob's arrival time T_2 is Exponential with expectation 1 hour.

(a) Determine c.

$$c\int_0^1 (t-t^2)\,dt = c\left(\tfrac{1}{2}-\tfrac{1}{3}\right) = c\cdot\tfrac{1}{6} \qquad \underline{\underline{c=6}}$$

(b) Determine the expectation and variance of T_1.

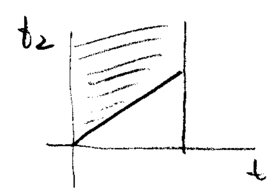
$$ET_1 = 6\int_0^1 t(t-t^2)\,dt = 6\int_0^1 (t^2-t^3)\,dt = 6\left(\tfrac{1}{3}-\tfrac{1}{4}\right) = \underline{\underline{\tfrac{1}{2}}}$$

$$ET_1^2 = 6\int_0^1 t^2(t-t^2)\,dt = 6\cdot\left(\tfrac{1}{4}-\tfrac{1}{5}\right) = \tfrac{3}{10}$$

$$\mathrm{Var}(T_1) = 0.3 - 0.25 = \underline{\underline{0.05}}$$

(c) Determine the probability that Alice arrives before Bob. Express the result as a single integral, which you do not need to compute.

$$f_{T_1,T_2}(t_1,t_2) = \begin{cases} 6(t_1-t_1^2)\,e^{-t_2} & t_1\in[0,1],\ t_2>0 \\ 0 & \text{otherwise} \end{cases}$$



$$P(T_1 < T_2) =$$

$$= \int_0^1 dt_1 \int_{t_1}^{\infty} 6(t_1-t_1^2)\,e^{-t_2}\,dt_2$$

$$= \int_0^1 6(t_1-t_1^2)\,e^{-t_1}\,dt_1$$

2. Roll a fair die 10 times. Let X be the number of 1's rolled, and Y the number of 2's rolled.
(a) Determine the joint probability mass function of X and Y. (Write a formula rather than a table.)

$$P(X=i, Y=j) = \frac{\binom{10}{i}\binom{10-i}{j} 4^{10-i-j}}{6^{10}}$$

$i, j = 1, \ldots, 10, \quad i+j \leq 10$

15

(b) Identify (as one of the well-known probability mass functions) the marginal probability mass function of X, and conditional probability mass function of X given $Y = 3$. Are X and Y independent?

X is Binomial $(10, \frac{1}{6})$

Given $Y=3$, X is Binomial $(7, \frac{1}{5})$. 10

So they are <u>not</u> independent.

3. At the entrance of a casino there is a bag, containing twenty balls, numbered 1 to 20. Upon arrival, each "guest" selects a ball from the bag three times, *with replacement*, and wins this *Bag Game* if all three selected numbers are the same. Assume that, starting tomorrow, 200 guests arrive each evening. Alice, the notorious gambler, is always among the 200.

(a) What is the probability that Alice wins the Bag Game tomorrow?

$$\frac{20}{20^3} = \frac{1}{400}$$

4

(b) Let N be the number of Bag Games Alice plays before she wins for the first time. (E.g., if Alice wins for the first time on the 78th evening, $N = 78$. Identify the distribution of N, and compute EN.

N is Geometric$\left(\frac{1}{400}\right)$, so $EN = 400$.

4

(c) Let W be the number of of winners tomorrow. Identify the distribution of W, and compute EW.

W is Binomial$\left(200, \frac{1}{400}\right)$ so $EW = \frac{1}{2}$.

$\approx$ Poisson$\left(\frac{1}{2}\right)$

4

(d) Approximate the probability that there will be at least 3 winners tomorrow.

$$P(W \geq 3) = 1 - P(W=0) - P(W=1) - P(W=2)$$
$$\approx 1 - e^{-1/2} - \frac{1}{2}e^{-1/2} - \frac{(1/2)^2}{2}e^{-1/2}$$
$$= 1 - \frac{13}{8}e^{-1/2}$$

5

(e) Approximate the probability that tomorrow there will be exactly 2 winners, among them Alice.

$$\approx \underset{\substack{\uparrow \\ \text{Alice wins}}}{\frac{1}{400}} \cdot \underset{\substack{\uparrow \\ \text{one additional winner}}}{\frac{1}{2}e^{-1/2}}$$

4

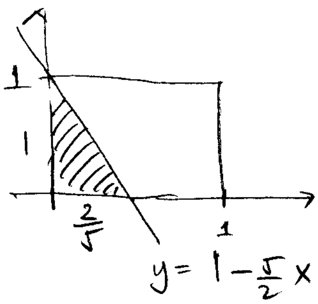
(f) Call an evening with exactly 3 winners a *Triple*. Alice will win a car if she is among the winners on the very first Triple evening. Determine (exactly, as a simple fraction) the probability that Alice wins the car.

$$\frac{3}{200}$$ (The 3 winners are 3 random people from 200)

4

4. Select a random point (X, Y) in the unit square $\{(x, y) : 0 \le x, y \le 1\}$. Call the selected point a *hit* if $5X + 2Y \le 2$.

(a) Compute the probability that the selected point is a hit. Draw a picture and give the answer as a simple fraction.



Area $=$ prob. $= \underline{\underline{\frac{1}{5}}}$. 10

(b) Now you select, independently, 40,000 such random points. Approximate the probability that the number of hits is at most 8,120. 15

The number of hits N is Binomial $(40{,}000, \frac{1}{5})$

$$P(N \le 8{,}120) = P\left(\frac{N - 8{,}000}{\sqrt{40{,}000 \cdot \frac{1}{5} \cdot \frac{4}{5}}} \le \frac{8{,}120 - 8{,}000}{\sqrt{}} \right)$$

$$\approx P\left(Z \le \frac{120}{200 \cdot 2 \cdot \frac{1}{5}} \right) = P\left(Z \le \frac{120}{80} \right) = P(Z \le 1.5)$$

$$\approx \underline{\underline{0.933}}$$

from table