Math 135A, Fall 2013.
Nov. 22. 2013.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____________________

ID#: ____________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. A random variable X has density function

$$f(x) = \begin{cases} c(2x+1) & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine c.

$$1 = c\int_0^1 (2x+1)\,dx = c \cdot (1 + 1)$$

$$\underline{\underline{c = \frac{1}{2}}}$$

(b) Compute Var(X).

$$EX = \frac{1}{2}\int_0^1 x(2x+1)\,dx = \frac{1}{2}\left(\frac{2}{3} + \frac{1}{2}\right) = \frac{7}{12}$$

$$E(X^2) = \frac{1}{2}\int_0^1 x^2(2x+1)\,dx = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right) = \frac{5}{12}$$

$$\mathrm{Var}(X) = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \underline{\underline{\frac{11}{144}}}$$

(c) Determine the probability density function of the random variable $Y = 1/\sqrt{X}$.

$$P(Y \le y) = P\left(\frac{1}{\sqrt{X}} \le y\right) = P\left(X \geq \frac{1}{y^2}\right)$$

$$y \in (1, \infty) \qquad = \frac{1}{2}\int_{1/y^2}^{\infty} (2x+1)\,dx$$

$$f_Y(y) = \frac{d}{dy}P(Y \le y) = -\frac{1}{2} \cdot \left(2 \cdot \frac{1}{y^2} + 1\right) \cdot \left(-\frac{2}{y^3}\right)$$

$$= \frac{y^2 + 2}{y^5} \qquad \text{for } y > 1$$

$$(0 \qquad \text{otherwise})$$

2. Roll a fair die 10 times. Let X be the total number of 1's rolled and Y the total number of 2's rolled.

(a) Identify the probability mass function of X and the probability mass function of Y.

Both Binomial $(10, \frac{1}{6})$.

(b) Write down the joint probability mass function of X and Y. (Write a formula rather than a table.)

$$P(X = x, Y = y) = \frac{\binom{10}{x}\binom{10-x}{y} 4^{10-x-y}}{6^{10}}$$

for $x, y = 1, \ldots, 10$, $x + y \leq 10$.

(c) Are X and Y independent? No $P(X = 10, Y = 1) = 0$, but $P(X = 10) \neq 0$, $P(Y = 1) \neq 0$.

(d) Compute the conditional probability $P(X = 1|Y = 8)$. Give the result as a simple fraction.

$$P(X = 1 | Y = 8) = \frac{2 \cdot 1 \cdot 4}{5^2} = \underline{\underline{\frac{8}{25}}}.$$

(Of the remaining two rolls, which are not 2, one must be 1 and one $3, \ldots, 6$.)

3. Each day, you perform the following random experiment with a bag that contains 100 balls, of which 2 are red. 48 are white and 50 are black. (These numbers are the same every day.) Select 2 balls at random from the bag *with* replacement and note their colors. Call the day *red* if both balls selected are red. and *black* if both balls selected are black.

(a) Write down the exact probability that at least two days among next 1000 days are red.

$$p = P(\text{red}) = \left(\frac{1}{50}\right)^2 = \frac{1}{2500}$$

Answer: $1 - P(\text{no red day}) - P(\text{one red day})$

$$= 1 - (1-p)^{1000} - 1000\, p\, (1-p)^{999}$$

(b) Using an appropriate approximation, estimate the probability in (a).

No. of red days is $\text{Binomial}\left(1000, \frac{1}{2500}\right) \approx \text{Poisson}\left(\frac{2}{5}\right)$.

Answer: $1 - e^{-2/5} - \frac{2}{5} e^{-2/5} = 1 - \frac{7}{5} e^{-2/5}$

(c) Using an appropriate approximation, estimate the probability that at least 84 among next 300 days are black.

X = no. of black days is $\text{Binomial}\left(300, \frac{1}{4}\right)$

$$P(X \geq 84) = P\left(\frac{X - 300\cdot\frac{1}{4}}{\sqrt{300\cdot\frac{1}{4}\cdot\frac{3}{4}}} \geq \frac{84 - 300\cdot\frac{1}{4}}{\sqrt{300\cdot\frac{1}{4}\cdot\frac{3}{4}}}\right)$$

Z is $N(0,1)$ $\qquad \approx P\left(Z \geq \frac{9}{\frac{3}{4}\cdot 10}\right)$

$$= P(Z \geq 1.2) = 1 - P(Z \leq 1.2)$$

$$\approx 1 - 0.885$$

$$= 0.115$$

4. Assume that Alice will arrive home this evening at a random time, uniformly distributed between 5 and 6 (all times p.m.). Bob promised to call Alice "sometime after 5," which in Bob's case means that he will, starting at 5, wait for an exponential amount of time with expectation 30 minutes and then call. Alice's arrival is independent of Bob's call.

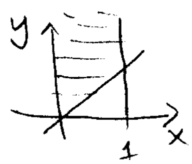
(a) Compute the probability that Alice will *not* miss Bob's call. Time unit = 1 hr

$X =$ Alice's arrival is Uniform on $[0,1]$ (started at 5)

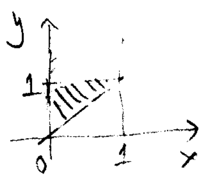
$Y =$ Bob's call has density $f_Y(y) = \begin{cases} 2e^{-2y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$$P(X \leq Y) = \int_0^1 dx \int_x^\infty 2e^{-2y}\,dy = \int_0^1 e^{-2x}\,dx = \underline{\underline{\tfrac{1}{2}(1-e^{-2})}}$$

. Joint density $f(x,y) = \begin{cases} 2e^{-2y} & , \; y \geq 0, \; 0 \leq x \leq 1 \\ 0 & , \; \text{otherwise} \end{cases}$



(b) Compute the conditional probability that, given that Alice will not miss the call. Bob will call before 6.



$$P(Y \leq 1 \mid X \leq Y) = \frac{P(Y \leq 1, X \leq Y)}{P(X \leq Y)}$$

$$= \frac{2}{1-e^{-2}} \int_0^1 dx \int_x^1 2e^{-2y}\,dy$$

$$= \frac{2}{1-e^{-2}} \int_0^1 \left(-e^{-2} + e^{-2x}\right) dx$$

$$= \frac{2}{1-e^{-2}} \left(-e^{-2} + \frac{1}{2} - \frac{1}{2}e^{-2}\right)$$

$$= \frac{1-3e^{-2}}{1-e^{-2}}$$