Math 135A, Winter 2013.
Mar. 6, 2013.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. Random variable X has density given by $f(x) = \begin{cases} c(\sqrt{x} - x) & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

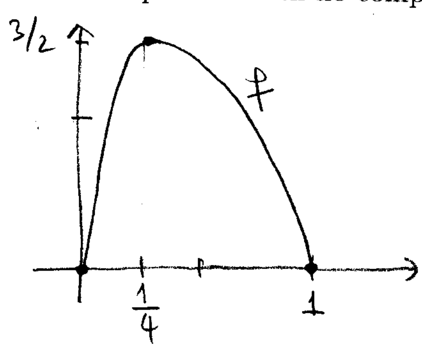
Random variable Y is independent of X and is uniformly distributed on $[0, 2]$.

(a) Determine c and compute EX. (Recall $\int_0^1 x^n\, dx = 1/(n+1)$ for $n > -1$.)

$$c \int_0^1 (\sqrt{x} - x)\, dx = 1, \quad c\left(\frac{2}{3} - \frac{1}{2}\right) = 1, \quad c \cdot \frac{1}{6} = 1, \quad \underline{\underline{c = 6}}.$$

$$EX = 6 \int_0^1 x(\sqrt{x} - x)\, dx = 6\left(\frac{2}{5} - \frac{1}{3}\right) = 6 \cdot \frac{1}{15} = \underline{\underline{\frac{2}{5}}}$$

(b) Let $\epsilon = 10^{-8}$. For a fixed real number a, let $\phi(a) = P(X \in [a, a + \epsilon])$. Rank by size the following four probabilities: $\phi(0)$, $\phi(1/4)$, $\phi(1/2)$, $\phi(1)$. (*Hint*. Sketch the graph of density f. After that, you can answer the question with no computation.)



$$f'(x) = 6\left(\frac{1}{2}x^{-1/2} - 1\right) = \frac{3(1 - 2\sqrt{x})}{\sqrt{x}} = 0 \text{ when } x = \frac{1}{4}$$

$\varphi(0) \approx \varepsilon$

$\varphi(\frac{1}{4}) \approx \frac{3}{2}\varepsilon$

$\varphi(\frac{1}{2}) \approx f(\frac{1}{2})\varepsilon, \quad f(\frac{1}{2}) < \frac{3}{2}$

$\varphi(1) = 0$

$$\varphi(1) < \varphi(0) < \varphi\left(\frac{1}{2}\right) < \varphi\left(\frac{1}{4}\right)$$

(c) Compute $P(Y \le 2X)$.

$$= \int_0^1 dx \int_0^{2x} 6(\sqrt{x} - x) \cdot \frac{1}{2}\, dy = 3\int_0^1 dx\ 2x \cdot (\sqrt{x} - x)$$

$$= \underline{\underline{\frac{2}{5}}} \quad (\text{same as } EX \text{ in (a)})$$

2. Toss a fair coin 10 times, with tosses numbered $1, \ldots, 10$. Let X be the number of Heads tossed. Let Y be 0 if there are no Heads and otherwise the number of the toss with the last Heads. (For example, if the outcome of tosses is THTHHTTTTT, then $X = 3$ and $Y = 5$.)

(a) Determine the joint probability function of X and Y. (Write a formula rather than a table.)

$$P(X = i, Y = j) = \begin{cases} \frac{1}{2^{10}} \cdot \binom{j-1}{i-1} & \text{if } i > 0, j \geq i \\ \frac{1}{2^{10}} & \text{if } i = 0, j = 0 \\ 0 & \text{otherwise} \end{cases}$$

$0 \leq i \leq j \leq 10$

(b) Are X and Y independent?

No $\quad P(X = 1, Y = 0) = 0$,

while $P(X = 1) \neq 0$, $P(Y = 0) \neq 0$.

3. In the *Four Dice* game, a dice is rolled four times. Alice bets that the four rolls result in the same number, while Bob bets that 6 is rolled on each of the first three rolls. Assume this game is played repeatedly (and independently) once a day, starting today.

(a) Let N_1 be the number of days until Alice wins for the first time (i.e., if she wins today, then $N_1 = 1$; if she loses today, but wins tomorrow, $N_1 = 2$, etc.). Compute EN_1.

N_1 is Geometric$\left(\frac{1}{6^3}\right)$, so $EN_1 = 6^3$.

(b) Using the appropriate approximation, estimate the probability that Alice wins at least 3 times in the first $432 = 2 \cdot 6^3$ days.

No. of times Alice wins is Binomial$\left(2 \cdot 6^3, \frac{1}{6^3}\right)$

$\approx$ Poisson(2).

<u>Answer</u>. $1 - e^{-2} - 2e^{-2} - \frac{2^2}{2} e^{-2} = \underline{\underline{1 - 5e^{-2}}}$

(c) Let N_1 be as in (a), and let N_2 be the number of days until Bob wins for the first time. Compute $P(N_1 < N_2)$. (*Hint.* Think of a game which Alice wins exactly when $N_1 < N_2$.)

<u>Game</u>: Alice wins if she wins her bet, and Bob does not.
Alice loses if Bob wins his bet.
Otherwise repeat.

$P(N_1 < N_2) = P$(Alice wins the game)

$= P$(A. wins the game | game is decided)

$$= \frac{\frac{5}{6^4}}{\frac{1}{6^3} + \frac{1}{6^3} - \frac{1}{6^4}} = \frac{5}{6+6-1} = \underline{\underline{\frac{5}{11}}}$$

P(game decided) = P({4 rolls same} ∪ {1st 3 rolls 6})

4. Now Alice plays the *Four Tosses* game. In this game, she tosses four coins and bets that all four tosses have the same outcome (i.e., that either all are Heads or all are Tails). Assume that she plays this game $448 = 7 \cdot 8 \cdot 8$ times (independently). Let X be the number of times she wins the bet.
(a) Identify (as one of the famous probability mass functions) the probability mass function of X. Compute EX and $\mathrm{Var}(X)$.

X is Binomial$(448, \frac{1}{8})$

$$EX = 448 \cdot \frac{1}{8} = \underline{\underline{56}}, \quad \mathrm{Var}(X) = 448 \cdot \frac{1}{8} \cdot \frac{7}{8} = \underline{\underline{49}}$$

(b) Using a relevant approximation, estimate the probability that X is at least ~~65~~ 64. Give the result as a decimal number. (Use ~~1/7 ≈ 0.29~~.) $1/7 \approx 0.14$

$$P(X \geq 65) = P\left(\frac{X-56}{\sqrt{49}} \geq \frac{64-56}{\sqrt{49}}\right)$$

$$\approx P(Z \geq \tfrac{8}{7}) \approx 1 - P(Z \leq 1.14)$$

$Z \sim N(0,1)$

$$\approx 1 - 0.873$$

$$= \underline{\underline{0.127}}$$