Math 135B, Spring 2011.
June 8, 2011.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____________________

ID#: ____________________

Instructions: Each of the 6 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Calculators, books or notes are not allowed. Unless you are directed to do so, or it is required for further work, do *not* evaluate complicated expressions to give the result as a decimal number.

Make sure that you have a total of 8 pages (including this one) with 6 problems.

1	
2	
3	
4	
5	
6	
TOTAL	

1. Suspects $1, 2, \ldots$ are coming before a certain court. They are associated with i. i. d. random variables $X_1, X_2, \ldots$, continuously distributed with density $f(x) = \begin{cases} 2x & \text{if } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$ Given $X_i = x$, suspect i is *guilty* with probability x (and *innocent* otherwise); one may imagine that guilt is determined by flip of a coin with Heads probability x. The court knows each X_i (the strength of evidence against i), but not whether any suspect is guilty (i.e., the coin flip is not observed). The court *convicts* some suspects based on the X_i and a fixed parameter $\alpha > 0$. (Note that convicted and guilty have different meanings.)

(a) Compute the long-term proportion of innocent people before the court.

By weak law of large numbers, and as suspects are innocent independently, this is given by

$$P(X_i \text{ innocent}) = 1 - P(X_i \text{ guilty})$$
$$= 1 - \int_0^1 x \cdot 2x\,dx = 1 - \frac{2}{3} = \underline{\underline{\frac{1}{3}}}$$

13

(b) Assume the court convicts the suspect i exactly when $X_i > \alpha$. Compute the long-term proportion of innocent people among those that are convicted.

Again by w.l.l.n., this is given by

$$P(X_i \text{ innocent} \mid X_i > \alpha) = \frac{P(X_i \text{ inn.}, X_i > \alpha)}{P(X_i > \alpha)}$$

$$= \frac{\int_\alpha^1 (1-x)\,2x\,dx}{\int_\alpha^1 2x\,dx} = \frac{1-\alpha^2 - \frac{2}{3}(1-\alpha^3)}{1-\alpha^2}$$

12

$$= \frac{1 - 3\alpha^2 + 2\alpha^3}{3(1-\alpha^2)} = \underline{\underline{\frac{1+\alpha-2\alpha^3}{3(1+\alpha)}}}$$

1, continued.

(c) *Extra credit, only attempt if you have time. You will receive zero credit unless your solution is correct and fully justified.* Now assume that suspect i may be convicted in two ways: *by evidence*, i.e., if $X_i > \alpha$, or *by hysteria*. For $i \geq 2$, suspect i is convicted by hysteria if $i-1$ is convicted by evidence; on the other hand, if $i-1$ is not convicted or convicted by hysteria, then i can only be convicted by evidence (i.e., is convicted if and only if $X_i > \alpha$). Suspect 1 can only be convicted by evidence. For example, if $\alpha = 0.5$ and first four X_i are 0.3, 0.6, 0.7, 0.2, then the convicted suspects are 2 (by evidence) and 3 (by hysteria), and this would not change if X_3 changed to 0.1. Again, answer the question in (b).

Two state MC: $0 \to 1$ w.p. $P(X_1 > \alpha) = 1 - \alpha^2$

$0 \to 0$ w.p. α^2

$1 \to 0$ w.p. 1

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \alpha^2 & 1-\alpha^2 \\ 1 & 1 & 0 \end{array} \text{ ; inv. distr. } \left[\frac{1}{2-\alpha^2}, \frac{1-\alpha^2}{2-\alpha^2}\right].$$

When the MC is in state 0, evidence is evaluated; when it is in state 1, it is not and the suspect is convicted.

prop. of convicted: $\pi_0 \cdot P(X_1 > \alpha) + \pi_1$

$$= \frac{1}{2-\alpha^2} \cdot (1-\alpha^2) + \frac{1-\alpha^2}{2-\alpha^2} = \frac{2(1-\alpha^2)}{2-\alpha^2}$$

prop. of innocent & convicted: $\pi_0 \cdot P(X_1 > \alpha, \text{inn.}) + \pi_1 P(\text{inn.})$

$$= \frac{1}{2-\alpha^2} \cdot \frac{1-3\alpha^2+2\alpha^3}{3} + \frac{1-\alpha^2}{2-\alpha^2} \cdot \frac{1}{3}$$

Answer: $$\frac{1-3\alpha^2+2\alpha^3+1-\alpha^2}{6(1-\alpha^2)} = \frac{2-4\alpha^2+2\alpha^3}{6(1-\alpha^2)}$$

$$= \frac{1+\alpha-\alpha^2}{3(1+\alpha)}$$

(No longer 0 when $\alpha = 1$!)

2. Initially (at 0th step), Alice has 3 cards and Bob none. Here are the rules at each step of the game. If one of them has no cards, then *redistribution* occurs: each card is given independently to Alice or Bob, with probability $\frac{1}{2}$. If both have at least one card, then one of the three cards is chosen at random; if the chosen card is in Alice's possession it is given to Bob, and if it is in Bob's possession it is given to Alice.

(a) Determine the transition probability matrix of the Markov chain whose state is the number of cards Alice has.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/8 & 3/8 & 3/8 & 1/8 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/8 & 3/8 & 3/8 & 1/8 \end{bmatrix} \end{array}$$

9

(b) Write down an expression for the probability that Alice has all the cards after 5 steps, but no cards after 10 steps.

$$P_{33}^5 \; P_{30}^5$$

4

(c) Compute the invariant distribution for this chain (you may use symmetry to shorten your computations).

$$[\pi_0 \quad \pi_1 \quad \pi_2 \quad \pi_3]P = [\pi_0 \quad \pi_1 \quad \pi_2 \quad \pi_3]$$

$\pi_0 = \pi_3$, $\pi_1 = \pi_2$ as A. & B. appear symmetrically

$$\tfrac{1}{8}\pi_0 + \tfrac{1}{3}\pi_1 + \tfrac{1}{8}\pi_0 = \pi_0$$

8

$$\tfrac{1}{3}\pi_1 = \tfrac{3}{4}\pi_0, \quad \pi_1 = \tfrac{9}{4}\pi_0$$

$$\pi_0 + \tfrac{9}{4}\pi_0 + \tfrac{9}{4}\pi_0 + \pi_0 = 1, \quad \pi_0 = \tfrac{4}{26}$$

Inv. distr. : $\left[\frac{4}{26}, \frac{9}{26}, \frac{9}{26}, \frac{4}{26}\right]$

(d) Compute the proportion of steps at which redistribution occurs.

4

$$\pi_0 + \pi_3 = \frac{8}{26} = \frac{4}{13}$$

3. Two random walkers walk on nonnegative integers. Alice starts at 0, while Bob starts at some large $n \geq 0$. Each second, each walker jumps to the right (i.e., adds a random number to the current position): Alice jumps by 2 or 3, each with probability 1/2; Bob jumps by 0 (probability $1-p$), 1 (probability $p/2$), or 2 (probability $p/2$). Let p_n be the probability that the two ever meet (i.e., occupy the same point at some time).

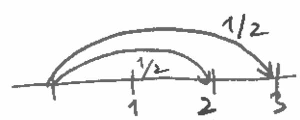
(a) Assume that $p = 0$, so that Bob never moves. Compute $\lim_{n\to\infty} p_n$.

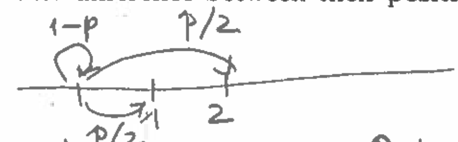
$$E[A\text{'s jump}] = \frac{1}{2}(2+3) = \frac{5}{2}$$

12

<u>Answer: $\frac{2}{5}$</u>

(b) Now assume that $p > 0$, and compute $\lim_{n\to\infty} p_n$. (*Hint*: difference between their ~~positions~~ jumps.)

A: 

B: 

Assume the two walks start from 0:

P(A's walk − B's walk ever hits n) 13

= P(A's walk ever hits B's walk + n)

A's walk − B's walk add each time:

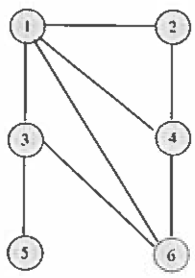
$$\begin{cases} 0 & \text{w.p. } \frac{1}{2}\cdot\frac{p}{2} = \frac{p}{4} \\ 1 & \text{w.p. } 2\cdot\frac{1}{2}\cdot\frac{p}{2} = \frac{p}{2} \\ 2 & \text{w.p. } \frac{1}{2}(1-p) + \frac{1}{2}\cdot\frac{p}{2} = \frac{1}{2} - \frac{p}{4} \\ 3 & \text{w.p. } \frac{1}{2}(1-p) = \frac{1}{2} - \frac{p}{2} \end{cases}$$

$$E[\text{jump} \mid \text{jump} > 0] = \frac{\frac{p}{2} + 2\left(\frac{1}{2} - \frac{p}{4}\right) + 3\left(\frac{1}{2} - \frac{p}{2}\right)}{1 - \frac{p}{4}} = \frac{2(5-3p)}{4-p}$$

<u>Answer: $\frac{4-p}{2(5-3p)}$</u>

4. A random walker is in one of the six vertices, labeled 1, 2, 3, 4, 5, and 6, of the graph in the picture. At each time, she moves to a randomly chosen vertex connected to her current position by an edge. (All choices are equally likely and the walker never stays at the same position for two successive steps.)

(a) Compute the (long-run) proportion of time the walker spends at each of the six vertices. Does this proportion depend on the walker's starting vertex?



Long-term proportions are given by

$$[\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6] = \left[\frac{4}{16}, \frac{2}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{16}, \frac{3}{16}\right]$$

(indep. of initial vertex)

(b) Compute the proportion of time the walker is at 1, and then at 2 next time.

$$\frac{4}{16} \cdot \frac{1}{4} = \underline{\underline{\frac{1}{16}}}$$

(c) Assume that the walker starts at vertex 1. What is the expected time before the walker returns to 1?

$$\frac{16}{4} = \underline{\underline{4}}$$

5. In a branching process, an individual has no descendants (in the next generation) with probability $1-p$, 1 descendant with probability $p/2$. and 3 descendants with probability $p/2$. The process starts with a single individual at generation 0. (Note that both answers below will depend on p.)

(a) Compute the probability π_0 that the process ever goes extinct.

$$\mu = EX_1 = \frac{p}{2} + \frac{3p}{2} = 2p$$

If $p \le \frac{1}{2}$, $\pi_0 = 1$,

If $p > \frac{1}{2}$:

$$\varphi(s) = 1-p+s\cdot\frac{p}{2} + s^3\cdot\frac{p}{2} = s$$

$$ps^3 + (p-2)s + 2(1-p) = 0$$

$$(s-1)(ps^2 + ps + 2(1-p)) = 0$$

$$\pi_0 = \frac{-p+\sqrt{p^2+8p(1-p)}}{2p} = \frac{-p+\sqrt{8p-7p^2}}{2p}$$

(b) Compute the expected population size of the third generation.

$$\mu^3 = (2p)^3$$

6. A sensitive instrument is put on line for a time period T (in hours). Two types of events, which occur as independent Poisson processes, may cause the instrument to malfunction: type A events occur at rate λ_A and type B events at rate λ_B.

(a) Assume that (1) $T = 2$ and that (2) 2 or more events of either type in time interval $[0, T]$ will cause the malfunction. What is the probability of malfunction?

P(malf.) = 1 − P(0 or 1 events in the combined A, B process)

$$= 1 - e^{-2(\lambda_A+\lambda_B)} - 2(\lambda_A+\lambda_B)\, e^{-2(\lambda_A+\lambda_B)}$$

(b) Assume that T is an Exponential random variable with $ET = 2$, and keep the assumption (2) from (a). What is the probability of malfunction?

P(malf.) = P(2 events in combined A, B-process, before 1 in T-process)

Embed T into a "T-process"

$$= \left(\frac{\lambda_A+\lambda_B}{\lambda_A+\lambda_B+1/2}\right)^2$$

(c) Assume (1) and (2) from (a). Given that exactly 1 type A event has occurred in time interval $[0, 1]$, what is the (conditional) probability of malfunction?

1 − P(no A-events in [1, 2]; no B-events in [0, 2])

$$= 1 - e^{-\lambda_A} \cdot e^{-2\lambda_B}$$

(d) Assume that T is an Exponential random variable with $ET = 2$, and the malfunction will happen if either at least 2 events of type A occur in $[0, T]$, or at least 1 event of type B occurs in $[0, T]$. What is the probability of malfunction?

Combine A, B, T processes. Malfunction occurs when either: (1) neither of first two events is T or (2) first two events are BT.

Answer: $\left(\frac{\lambda_A+\lambda_B}{\lambda_A+\lambda_B+1/2}\right)^2 + \frac{\lambda_B}{\lambda_A+\lambda_B+\frac{1}{2}} \cdot \frac{\frac{1}{2}}{\lambda_A+\lambda_B+\frac{1}{2}}$