Math 135B, Winter 2023.
Mar. 24, 2023.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 6 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work to receive full credit.* Clarity of your solutions may be a factor when determining credit. Electronic devices, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions: proper interpretation of exam questions is a part of the exam.

Make sure that you have a total of 7 pages (including this one) with 6 problems.

1	
2	
3	
4	
5	
6	
TOTAL	

1. A casino offers the following game. First, three fair coins are tossed, and the resulting number S of Heads is your *mark*. Once the mark is determined, it does not change throughout the game. A casino also has a random number generator, which on each call generates a *score*, an independent real random number, uniform in $[0,4]$. This generator is called repeatedly, generating scores until one of them exceeds your mark, at which point the game ends. Let $N \geq 1$ be the number of scores generated. For example, if $S=0$, then a single score will suffice and $N=1$, but if say $S=2$ and scores are 1.7, 0.4, 2.9, then $N=3$. Give all numerical answers below as simple fractions.

(a) Conditioned on $S=s$, what is the distribution of N? Compute EN.

Geometric$\left(\frac{4-s}{4}\right)$

$$EN = \sum_{s=0}^{3} \frac{4}{4-s}\binom{3}{s}\cdot\frac{1}{8}$$
$$= \frac{1}{2}\left(\frac{1}{4}+1+\frac{3}{2}+1\right)$$
$$= \underline{\underline{\frac{15}{8}}}$$

(b) Assume that the casino pays you the difference between your last score (i.e., the one that exceeds the mark) and the mark. This is your revenue R in the game. (In the above example, with $S=2$ and scores 1.7, 0.4, 2.9, you get paid 0.9.) Compute the expected revenue ER.

Conditioned on $S=s$, the last score is uniform on $[s,4]$

so $E[R|S=s] = \frac{s+4}{2} - s = \frac{4-s}{2}$.

So, $ER = \sum_{s=0}^{3} E[R|S=s]\,P(S=s) = \sum_{s=0}^{3} \frac{4-s}{2}\,P(S=s)$
$$= -\frac{1}{2}ES + 2 = -\frac{3}{4}+2 = \underline{\underline{\frac{5}{4}}}$$

(c) Assume that, for every score that is smaller than the mark (i.e., every score but the last), you have to pay the casino *four times* the difference between the mark and the score, which is your cost C of the game. (In the example in (b), the cost is $4(2-1.7)+4(2-0.4)$.) Determine the expected cost EC. With this cost, and the revenue as in (b), is this a favorable game for you?

Conditioned on $S=s$, the non-last scores are uniform on $[0,s]$. So $E[C|S=s] = 4\,E(N-1|S=s)\cdot\frac{s}{2}$

$(s=-(4-s)+4) \rightarrow \quad = 2s\cdot\left(\frac{4}{4-s}-1\right)$
$$= -8 - 2s + \frac{32}{4-s}$$

So, $\underline{\underline{EC}} = -8 - 2ES + 8EN = -8-3+15 = \underline{\underline{4}}$

$E[R-C] = \frac{15}{4} - 4 = -\frac{11}{4} < 0$, an <u>unfavorable game</u>.

2. Alice is browsing the internet. Each minute, she is on one of 4 web pages, call them pages 1, 2, 3, and 4. She starts at page 2, at minute 0. Each minute: if she is at (her favorite) page 1, she stays there with probability 0.7 and moves to one of the other three pages (2, 3, or 4) with equal probability 0.1 for each. If she is on one of the pages $k \in \{2,3,4\}$, she stays on the same page k with probability 0.4, and moves to one of the other 3 pages (in $\{1,2,3,4\} \setminus \{k\}$) with equal probability 0.2 for each.

(a) Determine the transition probability matrix of the Markov chain whose state is the web page Alice is browsing.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.7 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.4 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.4 \end{bmatrix} \end{array}$$

(b) Write down an expression for the probability that Alice is browsing web page 1 at minute 5 and then again is browsing page 1 at minute 8. Do not evaluate.

$$P^5_{21} \cdot P^3_{11}$$

(c) Compute the invariant distribution for this chain (you may use symmetry to shorten your computations).

$\pi_2 = \pi_3 = \pi_4$, by symmetry

$0.7\pi_1 + 0.2(\pi_1 + \pi_2 + \pi_3) = \pi_1$

$-0.3\pi_1 + 0.6\pi_2 = 0$, $\pi_1 = 2\pi_2$

$\pi_1 + 3\pi_2 = 1$

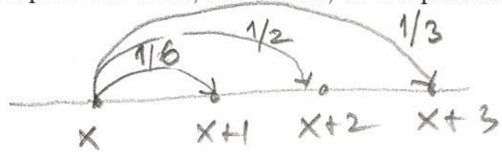
$5\pi_2 = 1$

$\pi_2 = \frac{1}{5} = \pi_3 = \pi_4$

$\pi_1 = \frac{2}{5}$

3. A particle moves on nonnegative integers, starting at 0. Each minute, the particle jumps to the right (i.e., adds a random number to the current position): it jumps by 1 with probability 1/6, by 2 with probability 1/2, and by 3 with probability 1/3.

(a) Compute the limit, as $n \to \infty$, of the probability p_n that the particle ever occupies n.



$$\mu = E[\text{jump}] = \frac{1}{6} + \frac{1}{2}\cdot 2 + \frac{1}{3}\cdot 3 = \frac{13}{6}$$

$$\lim_{n\to\infty} p_n = \frac{1}{\mu} = \underline{\underline{\frac{6}{13}}}$$

(b) Compute the limit, as $n \to \infty$, of the probability that the particle ever occupies both n and $n+1$.

$$\lim_{n\to\infty} p_n \cdot \frac{1}{6} = \underline{\underline{\frac{1}{13}}}$$

(c) Compute the expected number of jumps that the particle needs to reach a nonzero multiple of 3 for the first time.

Let X_n be the position of the walk <u>mod</u> 3. This is a M.C. on $\{0,1,2\}$ with transition matrix:

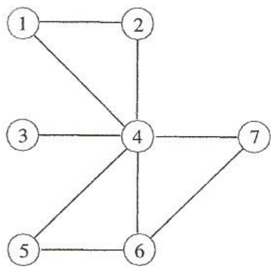
$$P = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/6 & 1/2 \\ 1/2 & 1/3 & 1/6 \\ 1/6 & 1/2 & 1/3 \end{bmatrix},$$

which is doubly stochastic, with invariant distribution $\pi = \left[\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\right]$. We are looking for the expected return time to 0, so the answer is $\frac{1}{\pi_0} = \underline{\underline{3}}$.

4. A particle is in one of the seven vertices, labeled 1, ..., 7, of the graph in the picture. At each time, it moves to a randomly chosen vertex connected to its current position by an edge. (All choices are equally likely and the walker never stays at the same position for two successive steps.)

(a) Compute the (long-term) proportion of time the walker spends at each of the seven vertices. Does this proportion depend on the walker's starting vertex?

Reversible distribution:



$\pi_1 = \frac{2}{18}$

$\pi_2 = \frac{2}{18}$

$\pi_3 = \frac{1}{18}$

$\pi_4 = \frac{6}{18}$

$\pi_5 = \frac{2}{18}$

$\pi_6 = \frac{3}{18}$

$\pi_7 = \frac{2}{18}$

These are respective proportions, which do not depend on the starting vertex.

(b) Compute the proportion of time the walker is at an odd vertex (i.e., at a vertex in $\{1, 3, 5, 7\}$), and then at vertex 4 at the next time.

$$\pi_1 \cdot \frac{1}{2} + \pi_3 + \pi_5 \cdot \frac{1}{2} + \pi_7 \cdot \frac{1}{2}$$

$$= 3 \cdot \frac{1}{2} \cdot \frac{2}{18} + \frac{1}{18} = \frac{2}{9}$$

(c) Assume that the walker starts at vertex 1. What is the expected time before the walker returns to 1?

$$\frac{1}{\pi_1} = 9$$

5. In a branching process X_n, an individual has no descendants (in the next generation) with probability 1/6, 1 descendant with probability 1/2, 2 descendants with probability 1/6 and 3 descendants with probability 1/6. The process starts with a single individual at generation 0, that is, $X_0 = 1$.
(a) Compute the probability π_0 that the process dies out. (Note: $s^3+s^2-3s+1 = (s-1)(s^2+2s-1)$.)

$$\varphi(s) = \frac{1}{6} + \frac{1}{2}s + \frac{1}{6}s^2 + \frac{1}{6}s^3$$

$$\varphi(s) = s$$

$$1 + 3s + s^2 + s^3 = 6s$$

$$s^3 + s^2 - 3s + 1 = 0$$

$$(s-1)(s^2+2s-1) = 0$$

$$s = \frac{-2 \pm \sqrt{8}}{2} = -1 + \sqrt{2} < 1$$

So, $\underline{\underline{\pi_0 = -1 + \sqrt{2}}}$.

(b) Write down an expression for the probability that the process dies out at or before generation 3, that is, $P(X_3 = 0)$. Do not simplify.

Answer: $\varphi(\varphi(\varphi(0))) = \varphi(\varphi(\frac{1}{6}))$

$$= \varphi\left(\frac{1}{6} + \frac{1}{2}\cdot\frac{1}{6} + \frac{1}{6^3} + \frac{1}{6^3}\right)$$

(c) Now change the distribution of descendants as follows: an individual has no descendants with probability 1/2, 1 descendant with probability 1/6, 2 descendants with probability 1/6 and 3 descendants with probability 1/6. For this new branching process, compute EX_1, EX_{10}, and the probability that it dies out.

$\mu = \underline{\underline{EX_1}} = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} = \underline{\underline{1}}$, so $\underline{\underline{EX_{10} = 1}}$,

and $\underline{\underline{\pi_0 = 1}}$, as the process with $\mu \le 1$ dies out with probability 1.

6. An online store has just opened for business. It sells two items, A and B. Orders for the two items come as two independent Poisson processes: for item A they come at the rate λ_A and for item B at the rate λ_B. Assume the time unit is an hour.

(a) What is the probability that there will be no orders (for either item) in the first two hours?

Combined process has rate $\lambda_A + \lambda_B$.

Answer: $e^{-2(\lambda_A+\lambda_B)}$

(b) What is the probability that the first three orders the store receives will all be for item A?

Allocate A's with prob. $\frac{\lambda_A}{\lambda_A+\lambda_B}$ to the events in the combined process.

Answer: $\left(\frac{\lambda_A}{\lambda_B+\lambda_B}\right)^3$

(c) Bob is processing orders but he will go to lunch at a random time which is an Exponential random variable with expectation 4 (and independent of the order processes). What is the probability of the event that at least one order for A and at least one order for B will arrive before Bob's lunch?

Three indep. processes A, B, L with rates $\lambda_A, \lambda_B, \frac{1}{4}$.

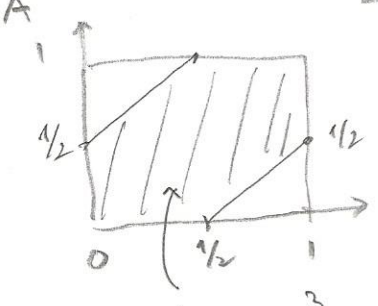
P(1st event A, then B before L) + P(1st event B, then A before L)

$$= \frac{\lambda_A}{\lambda_A+\lambda_B+\frac{1}{4}} \cdot \frac{\lambda_B}{\lambda_B+\frac{1}{4}} + \frac{\lambda_B}{\lambda_A+\lambda_B+\frac{1}{4}} \cdot \frac{\lambda_A}{\lambda_A+\frac{1}{4}}$$

(d) Compute that probability that, within the next hour, the following event happens: there will be exactly one order for A and exactly one order for B and that these two orders will be made within half an hour of each other.

Answer: $\underbrace{\lambda_A \cdot e^{-\lambda_A}}_{1\,A} \cdot \underbrace{\lambda_B \cdot e^{-\lambda_B}}_{1\,B} \cdot \underbrace{\frac{3}{4}}$

the two events are indep and unif. in $[0,1]$; so this is $P(|U_1-U_2| \le 1/2)$ where (U_1, U_2) is a random pt. in $[0,1]^2$.



Area $= \frac{3}{4}$