Math 135B, Winter 2023.

Homework 1

Due: Jan. 20, 2023

1. (a) Assume that X_n are random variables such that that $X_n \to a$ in probability, where a is a (nonrandom) constant. Suppose also $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Show that $f(X_n) \to f(a)$ in probability.

(b) Assume that X_n and Y_n are random variables such that X_n and Y_n are defined on the same probability space (so that they can be added). Assume also that $X_n \to a$ and $Y_n \to b$ in probability, where a and b are (nonrandom) constants. Show that $X_n + Y_n \to a + b$ in probability.

2. Assume that $X_1, X_2, \dots$ are independent random variables, all uniform on $[0, 1]$. Compute the limit, in probability, of the following random variables:

(a) $\frac{1}{n}\sum_{i=1}^{n} X_n$;

(b) $\frac{1}{n}\sum_{i=1}^{n} X_n^2$;

(c) $\frac{1}{n}\sum_{i=1}^{n} X_i X_{i+1}$; and

(d) $(X_1 \cdots X_n)^{1/n}$.

3. Assume you have $2n$ cards with n colors, with 2 cards of each color. Select n cards without replacement, and let N_n be the number of colors that *are not* represented in your selection.

(a) Compute EN_n and $\mathrm{Var}(N_n)$.

(b) Determine a constant c so that $\frac{1}{n}N_n \to c$, in probability.

(c) Let M_n be the the number of colors that *are* represented in your selection. Determine a constant d so that $\frac{1}{n}M_n \to d$, in probability.

4. (From a Final Exam at Queen's University, Ontario.) An urn contains m red and n blue balls. Balls are drawn one at a time withour replacement until all m red balls are drawn. Let T be the number of draws required. Compute ET. (*Hint*. The best way is to relate T to the number N of blue balls that *remain* in the urn after all red balls are drawn.)

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Homework 2

Due: Jan. 27, 2023

1. Assume that $X_1, \ldots, X_5$ are independent random variables with the same density

$$f(x) = \begin{cases} \frac{1}{e-1} e^x & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the moment generating function of X_1.

(b) Compute the moment generating function of $S = X_1 + \ldots + X_5$.

(c) Compute the moment generating function of $Y = X_1 + 2X_2$.

2. Assume that X is chosen at random from numbers $-1, 0, 1$, each with equal probability.

(a) Compute the moment generating function of X.

(b) Let $X_1, X_2, \ldots$ be independent and all distributed as X, and let $S_n = X_1 + \ldots + X_n$. Show that, for every $\epsilon > 0$, $P(S_n \geq \epsilon n)$ and $P(S_n \leq -\epsilon n)$ are for large n smaller that n^{-10}.

(c) Let $X_1, X_2, \ldots$ be as in (b) and let M_n be the maximal absolute value of the sum of some n consecutive terms of $X_1, \ldots, X_{n^2}$. Show that $M_n/n \to 0$ in probability.

3. (I got this problem from a high-school student. This is a harder problem, and you do not have to turn it in.) The median of a sequence of $2n+1$ numbers is the element a of the sequence such n other elements are at least a and n other elements are at most a; that is, it is the middle number after the sequence is ordered. Roll a fair die $2n+1$ times and let M_n be the median of the numbers rolled. Approximate EM_n^2 for large n and find an upper bound for the error in your approximation. (*Hints.* The distribution of M_n is symmetric. With high probability, M_n is 3 or 4. Use Problem 4 in Chapter 10.)

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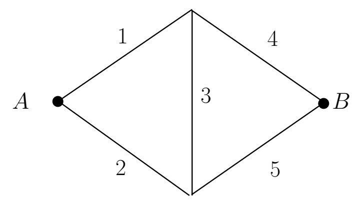
Homework 3

Due: Mar. 3, 2023

1. The road system between towns A and B is as in the picture. Assume that road i is open with probability p_i and that the roads are open independently. Compute the probability that A and B are connected.



2. In a sequence of n Bernoulli trials, each trial is independently a success with probability $p \in (0,1)$ and failure with probability $1-p$. Let p_n be the probability that there are no consecutive failures.

(a) Compute p_0, p_1, and p_2.

(b) Give the recursive equation, expressing p_n by p_{n-1} and p_{n-2}. Write down the general solution to the recursion and explain how you would determine the two constants. You do not need to compute the constants by hand.

(c) Compute $\lim_{n\to\infty}(p_{n+1}/p_n)$.

Here is a reminder on how to solve a second order linear recursion, given by $x_n = ax_{n-1} + bx_{n-2}$, where $a, b \in \mathbb{R}$. The characteristic equation $\lambda^2 = a\lambda + b$ either has two distinct solutions $\lambda_1 \neq \lambda_2$, or a single solution λ_1. In the first case, the general solution is $x_n = A\lambda_1^n + B\lambda_2^n$ and in the second case it is $x_n = \lambda_1^n(A + Bn)$. The constants A and B can be determined if we know, for example, x_0 and x_1, which gives two linear equations with two unknowns.

3. Assume that T is an exponential random variable with $ET = 1$, which is the time when a lightbulb goes out. Given $T = t$, the waiting time X for a repairman is (a) uniform on $[t/2, t]$, (b) exponential with expectation $1/t$. In each case, compute EX. In case (b), also compute the conditional density of T given $X = x$.

4. (A job interview problem from Google.) You can roll a 6-sided die at most 3 times. If you roll x on the first roll, you can either accept x dollars, or forgo this amount (so you get nothing from this roll) and continue rolling. If you continue, the same rule applies for the second roll. If you get to the third roll, you get x dollars if your roll x on the third roll and the game stops. What is the best strategy for playing this game and what is the expected payoff?

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Homework 4

Due: Mar. 10, 2023

Note. You do not need to evaluate products of matrices by hand. Once you get your answer to the form in which the numerical answer can be obtained by a matrix computation, you may stop.

1. At each time $t = 0, 1, 2, \ldots$, each of the balls A and B is either in urn 1 or urn 2. At each time step, we select ball A with probability p and ball B with probability $1 - p$, and then keep the selected ball in the same urn with probability $1/3$ and place it in the other urn with probability $2/3$. Compute the probability that ball A is in urn 1 and ball B is in urn 2 after 6 time steps provided that: (a) initially, both balls are in urn 1; and (b) initially, each ball is independently put in one of the two urns with equal probability.

2. Assume that the Markov chain has states 1, 2, and 3 with the transition matrix

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \\ 1/3 & 1/2 & 1/6 \end{bmatrix}.$$

Assume $P(X_0 = 1) = P(X_0 = 2) = 1/6$. Compute the following quantities.

(a) $P(X_0 = X_1 = \cdots = X_{10} = 3)$.

(b) EX_3 and EX_3^2.

(c) EX_3X_4.

3. Alice has two coins: coin 1 has probability 0.7 of Heads and coin 2 has probability 0.6 of Heads. She starts by tossing coin 1 twice. From then on, Alice tosses coin 1 unless her previous two tosses were both Heads, in which case she tosses coin 2. Bob knows Alice's procedure, and is in addition only told that Alice's 10th toss was Heads. Help Bob determine the (conditional) probability that Alice used coin 1 on the 10th toss.

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Homework 5

Due: Mar. 17, 2023

Note. Again, do not need to do matrix computations by hand. I recommend you use MATLAB; if you would prefer not to, stop when your answer to the form in which the numerical answer can be obtained by a matrix computation.

1. Solve Problem 1 in Chapter 13 of the lecture notes.

2. A casino is offering the following game. There are three six-sided dice, a red, a green, and a blue. Five sides of each die have symbols $\mathcal{R}$, $\mathcal{G}$, $\mathcal{B}$, $+$, and $-$. The symbol on the sixth side depends on the die: on the red die it is $\mathcal{R}$ (so that the red die has two symbols $\mathcal{R}$); on the green die it is $\mathcal{B}$, and on the blue die it is $-$. If you roll $+$ or $-$, you respectively win or lose \$1 and the game is over. Otherwise, the symbol rolled determines the color of next die to roll ($\mathcal{R}$ for red, etc.). You continue rolling until your either roll $+$ or $-$.

(a) You are free to choose the die for your first roll. Determine which die you should choose. For this choice, compute your expected payoff and the expected number of die rolls.

(b) Now the casino changes the red die: the $-$ is replaced by $--$, and you lose \$2 if you roll $--$. Nothing else changes. Again, determine your optimal choice of the die, your expected payoff, and the expected number of die rolls in the game.

The way to approach this problem is as follows. Assume that you have a Markov chain whose first r states are absorbing and the other s states are transient. Then the transition matrix looks like this:

$$P = \begin{bmatrix} I_r & 0 \\ R & Q \end{bmatrix}.$$

Here, I_r is the $r \times r$ identity matrix, 0 is the $r \times s$ matrix of 0s, R is an $s \times r$ matrix, and Q is an $s \times s$ matrix. Suppose that S is the $s \times r$ matrix matrix whose entry S_{ij} is the probability that, starting from a transient state i, the chain ends up being absorbed into the absorbing state j. By conditioning on the first transition,

$$S_{ij} = R_{ij} + \sum_k Q_{ik} S_{kj},$$

where the sum is over all transient states k. (The above conditioning reflects that, in order to get to j, in a single step the chain either goes directly to j or to some transient state.) In the matrix form, we can write this $S = R + QS$, or

$$(I_s - Q)S = R.$$

Since we know that every entry of Q^n goes to 0 (by transience of the last s states), all eigenvalues of Q must be strictly less than 1 in absolute value. So, $I_s - Q$ must be invertible, and

$$S = (I_s - Q)^{-1} R.$$

Similarly, if N_{ij} is the expected total time, starting from a transient state i, that the process spends at a transient state j,

$$N_{ij} = \delta_{ij} + \sum_k Q_{ik} N_{kj}$$

(where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise), as the time spent at j is increased by 1 at the first step if $i = j$, but not if $i \neq j$. Letting N be the $s \times s$ matrix of these expected total times, we get $N = I_s + QN$, and so

$$N = (I_s - Q)^{-1}.$$

3. A random walker on $0, 1, 2, \ldots$ moves from each $i \geq 0$ either to $i+1$ or to 0 (and nowhere else). Assume the move to $i+1$ happens (1) with probability 1/2; (2) with probability $e^{-2^{-i}}$. Let X_n be its position after n steps.

(a) Show that the Markov chain X_n is irreducible.

(b) Assume that the walker starts at 0. Let R be the first time the chain returns to 0 (so that R is at least 1). Compute $P(R > n)$.

(c) Compute f_0, the probability of return to 0 in each case.

(d) Determine recurrence or transience of all states.

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Homework 6

Due: Feb. 24, 2023

1. Consider the branching process with the offspring distribution $p_0 = 3/32$, $p_1 = 15/32$, $p_2 = 9/32$, $p_3 = 5/32$. Let X_n be the population at generation n, with $X_0 = 1$.

(a) Find EX_5, the expectation and variance of the population at time 5.

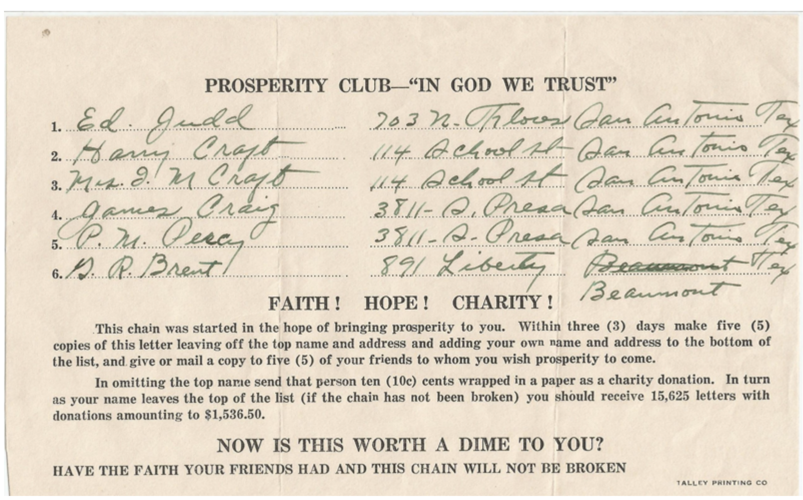
(b) Find expected combined population of generations 0 through 5.

(c) Find the probability that the process ever dies out.

(d) Find the expression for the probability that the process dies by time 3 (i.e., at time 3 or earlier).

(e) Find $E(X_5 X_6)$.

2. Send-a-Dime, which begun in Colorado in 1935, was one of the first popular chain letters. A recipient received a list of 6 names and addresses, in order from top the bottom. Also, the recipient was instructed to send a dime to the to the top address, make a new list with the top name left off, add own name and address at the bottom, then make 5 copies of the letter with the new list and send them to 5 friends.



PROSPERITY CLUB—"IN GOD WE TRUST"

1. Ed. Judd .... 703 N. Flores San Antonio Tex
2. Harry Craft .... 114 School St San Antonio Tex
3. Mrs. D. M. Craft .... 114 School St San Antonio Tex
4. James Craig .... 3811 S. Presa San Antonio Tex
5. P. M. Percy .... 3811 S. Presa San Antonio Tex
6. D. R. Brent .... 891 Liberty ~~Beaumont~~ Tex Beaumont

FAITH! HOPE! CHARITY!

This chain was started in the hope of bringing prosperity to you. Within three (3) days make five (5) copies of this letter leaving off the top name and address and adding your own name and address to the bottom of the list, and give or mail a copy to five (5) of your friends to whom you wish prosperity to come.

In omitting the top name send that person ten (10c) cents wrapped in a paper as a charity donation. In turn as your name leaves the top of the list (if the chain has not been broken) you should receive 15,625 letters with donations amounting to $1,536.50.

NOW IS THIS WORTH A DIME TO YOU?

HAVE THE FAITH YOUR FRIENDS HAD AND THIS CHAIN WILL NOT BE BROKEN

TALLEY PRINTING CO

As usual, the chain letter contains hints about how unwise it is for the recipient to "break the chain," that is, to not follow the instructions and discard the letter. Send-a-Dime generated hundreds of thousands of letters within a year, causing all sorts of problems for the USPS. Federal law, specifically CFR §353.8, prohibits chain letters and related pyramid schemes.

Assume that each recipient independently follows the instructions with probability p, and otherwise discards the letter without sending anything.

(a) Assume $p = 1$. Verify that the recipient indeed eventually receives the sum advertised in the letter.

(b) Determine the expected amount of money you receive, depending on p, and also variance of this amount. Compute the variance when $p = 1/5$.

(c) Determine, using the computer if necessary, the probability that the chain letter dies out for $p = 1/2$, $p = 1/5$ and $p = 1/10$. For these values of p, determine also the probability that the recipient receives at least one dime back.

3. (This is a discrete version of the M/M/1 queue, from the book by Grinstead and Snell. You do not need to turn in the solution to this problem.) Each minute, either 1 or 0 customers arrive to a server, based on a toss of the *arrival* coin with Heads probability p. Upon arrival, the customer either starts being served or joins the queue. When a customer is being served, the service is complete in that minute based on the independent *service* coin with heads probability r.

For example, if at some minute the queue is of size 4 and both coins come out Heads, then the queue is unchanged next time, but if only the service coin comes out Heads, the queue is reduced to 3. If the queue is empty, and a customer is being served, and both coins come out Heads, the situation is again the same the next minute, but now if only the service coin comes out Heads, the next minute nobody is served and the server has a free minute. In the latter situation, the service coin is immaterial, but if the arrival coin comes out Heads, in the next minute the queue is empty and a customer is being served. Note that every customer is served at least one minute.

At the start of the process (minute 0), there is a customer being served and an empty queue. Determine the probability that the server ever has a free minute.

(*Hints*. The service time is Geometric. Consider a branching process in which the offspring of a customer are the new customers that arrive during the service time. The generating function of the offspring distribution can be obtained by conditioning. Observe that the free minute happens if any only if the branching process dies out.)

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Homework 7

Due: Mar. 3, 2023

1. Show that the Markov chain in Problem 1 of Homework 4 has a unique invariant distribution and compute it. If, initially, both balls are in urn 1, approximate the probability that one of the urns is empty at time 1000.

2. Consider the Markov chain in Problem 2 of Homework 4.

(a) Show that it has a unique invariant distribution and compute it. What does the matrix P^n look like for large n?

(b) What proportion of time is the state of this chain strictly smaller than the previous state?

3. Consider the Markov chain in Problem 3 of Homework 4.

(a) Show that it has a unique invariant distribution and compute it.

(b) Compute the long-term proportion of tosses on which Alice uses coin 2.

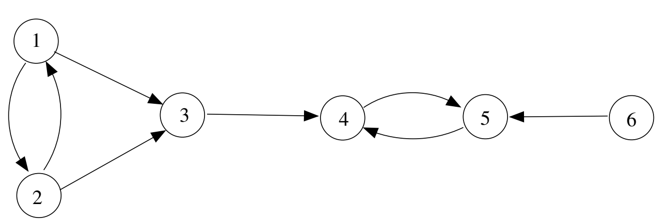
(c) Compute the long-term proportion of tosses on which Alice tosses Heads.

(d) Compute the long-term proportion of tosses on which Alice tosses Heads with coin 2.

(e) Bob pays Alice \$1 every time she tosses Heads with coin 1 and \$2 every time she tosses Heads with coin 2. Determine Alice's long-term average winnings per toss.

4. (A Wall Street job interview question.) You roll a fair six-sided die repeatedly and sum the numbers you roll. What is the expected number of rolls until the sum is a nonzero multiple of 4 for the first time?

5. Currently, it is not public knowledge how Google ranks the web pages in its search algorithm. The initial approach by S. Brin and L. Page in the mid 1990s, known as the *PageRank* algorithm, was using Markov chains for this purpose. Form an oriented graph, with N vertices, which are the web pages, in which an oriented edge goes from a web page x to a web page y if x contains a hyperlink to y. This creates a graph as in the following example with $N = 6$.



A random surfer begins at some page and then at every step chooses one of the outgoing arrows from the current page uniformly at random and then follows it to the next page; if there are no such such arrows, the surfer chooses one of the N pages uniformly at random and moves there. Let W be the transition matrix for the resulting Markov chain.

(a) Write down the matrix W for this example. Determine the classes and their recurrence or transience.

We see that the immediate problem is that the chain may not be irreducible. To deal with this problem, we introduce "damping." Pick a small $\epsilon > 0$. Then, the surfer follows the above algorithm with probability $1 - \epsilon$. Otherwise, with probability ϵ, it chooses a random state, i.e., evolves for one step according to the $N \times N$ transition matrix R that has all entries equal to $1/N$. A typical choice is $\epsilon = 0.15$; assume this for the rest of the problem. This gives the transition matrix

$$P = (1 - \epsilon)W + \epsilon R.$$

Using this transition matrix, the surfer's position has a unique invariant distribution, which is used to rank pages: the more often, in the long run, the surfer visits a web page x, the higher the rank of x.

(b) Rank the pages for the given example using this algorithm.

One problem with this version of *PageRank* is its vulnerability to the *Sybil attacks*. (The name comes from a famous, but possibly fictional, psychiatric case of a woman with multiple personality disorder.) That is, the owner of a page x can improve its rank by creating a number of new web pages $x_1, \ldots, x_k$, with links $x \to x_i$ and $x_i \to x$, for all $i = 1, \ldots, k$. In our example, say, the owner of page 5 creates a single new page 7 (so $k = 1$) with links $5 \to 7$ and $7 \to 5$ (but no other new links).

(c) Using *PageRank*, determine the new ranking of pages after the described Sybil attack in our example.

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Homework 8

Due: Mar. 10, 2023

Note. These three problems are adapted from job interviews on Wall Street. Assume that you may use a computer during the interview for coding (but not for looking for a solution on the internet). These seem difficult, albeit not impossible, problems to solve completely under pressure, so I suspect that the interviewers mainly want to gauge the candidate's reaction in such a situation. To get a bit of experience on how you might react, you are required, as a part of your submitted work, to write down an answer to each of the three problems *before you see the solution*. Each of your answers needs to at least outline an approach in a coherent language, and needs to contain a numerical answer, which can be just a guess if you are unable to make much progress; imagine that you are addressing your response to an interviewer. After that, you may verify and complete your work by consulting the solutions. (Of course, there is no penalty for incorrect preliminary answers.)

1. Alice has a coin with probability 0.6 of Heads, and Bob has coin with probability 0.3 of Heads. They toss a coin repeatedly: if the number of Heads tossed so far is odd, Alice tosses her coin, while if the number of Heads tossed so far is even, Bob tosses his coin. Let p_n be the probability of even number of Heads after n tosses. Approximate p_{1000}.

2. Two tokens are both initially positioned at 0 and move on nonnegative integers $0, 1, \dots$ as follows. There are two coins: the red coin is fair, with Heads probability $1/2$, while the blue coin has Heads probability $2/3$. Each minute, exactly one of the token makes a move based on the outcome of a toss of one of the coins: if the toss is Heads, the move is two unit steps to the right, and if the toss is Tails, the move is one unit step to the right. Here are the rules on which coin is used and which token makes the move:

- if both token occupy the same position, the red coin is used for the toss, and one of the tokens makes the move and the other stays put;
- otherwise, the blue coin is used for the toss, and the token that is behind (i.e., to the left) makes the move and the token that is ahead stays put.

Let p_n be the probability that the two tokens ever both simultaneously occupy n. Approximate p_{1000}.

3. A casino offers the following card game. A standard deck is shuffled and the dealer draws cards one by one, without replacement. You may ask the dealer to stop at any time. For each red card drawn, you win \$1; and for each black card drawn, you lose \$1. How much are you willing to pay to play the game?