Math 135B, Spring 2010.
April 21, 2010.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 4 problems is worth 25 points. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Calculators, books or notes are not allowed. Unless you are directed to do so, or it is required for further work, do *not* evaluate complicated expressions to give the result as a decimal number.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. Each minute i, $i = 1, 2, \ldots$, your score is independently chosen at random among the three numbers 1, 2, and 3. Call a minute $i \geq 2$ an *increase minute* if your score is strictly larger than the one at the previous minute (i.e., at minute $i - 1$). Let X_n, $n = 2, 3, \ldots$ be the number of increase minutes among the first n minutes.

(a) Compute EX_n.

$$I_i = I_{\{\text{min } i \text{ an increase minute}\}}$$

$$EI_i = P(1\underset{i}{\underset{\uparrow}{2}} \text{ or } 1\underset{i}{\underset{\uparrow}{3}} \text{ or } 2\underset{i}{\underset{\uparrow}{3}}) = \frac{3}{9} = \frac{1}{3}.$$

$$X_n = \sum_{i=2}^{n} I_i, \quad \text{so} \quad EX_n = \underline{\underline{(n-1) \cdot \frac{1}{3}}}.$$

(b) Compute $\mathrm{Var}(X_n)$.

$$\mathrm{Var}(X_n) = \sum_{i=2}^{n} \mathrm{Var}(I_i) + \sum_{\substack{i,j=2 \\ i \neq j}}^{n} \overset{\swarrow 0 \text{ unless } i \text{ and } j \text{ differ by } 1}{\mathrm{Cov}(I_i, I_j)}$$

$$= (n-1) \cdot \left(\frac{1}{3} - \left(\frac{1}{3}\right)^2\right) + 2 \sum_{i=2}^{n-1} \mathrm{Cov}(I_i, I_{i+1})$$

$$= (n-1) \cdot \frac{2}{9} + 2 \sum_{i=2}^{n-1} \left(E(I_i I_{i+1}) - \left(\frac{1}{3}\right)^2\right)$$

$$EI_i I_{i+1} = P(\text{both } i, i+1 \text{ increase minutes})$$

$$= P(12\underset{i+1}{\underset{\uparrow}{3}}) = \frac{1}{27}$$

$$= (n-1) \cdot \frac{2}{9} + 2 \cdot (n-2) \cdot \frac{-2}{27} = \frac{2}{27}(3(n-1) - 2(n-2))$$

$$= \underline{\underline{\frac{2}{27}(n+1)}}$$

(c) Determine the limit in probability, as $n \to \infty$, of $\frac{X_n}{n}$. Carefully explain your reasoning.

Let $Y_n = \frac{X_n}{n}$. then $EY_n = \frac{1}{3}\frac{n-1}{n} \to \frac{1}{3}$ and $\mathrm{Var}(Y_n) = \frac{2}{27}\frac{n+1}{n^2} \to 0$, as $n \to \infty$.

It follows that $\underline{\underline{Y_n \to \frac{1}{3}}}$ in probability.

2. Let U be a random variable, Uniform on $[0,1]$. Assume that $U_1, U_2, \ldots$ are independent and also Uniform on $[0,1]$, and let $S_n = U_1 + \cdots + U_n$.
(a) Compute the moment generating function of U.

$$\varphi_U(t) = E[e^{tU}] = \int_0^1 e^{tx}\,dx = \frac{1}{t} e^{tx}\Big|_0^1 = \frac{e^t - 1}{t}$$

(b) Compute the moment generating function of S_n.

$$\varphi_{S_n}(t) = \left(\frac{e^t - 1}{t}\right)^n$$

(By independence.)

(c) Explain how would you find an upper bound for the probability that S_n is larger than $0.75n$. (Do not attempt to carry out the computation.) Is this probability larger or smaller than $\frac{1}{n^4}$, for a very large n?

$$P(S_n \geq 0.75n) \leq e^{-I(0.75)n}$$

where $I(0.75) = \max_{t>0} \{0.75t - \log \varphi_U(t)\}$

As $0.75 > 0.5 = EU$, $I(0.75) > 0$ and so

$P(S_n \geq 0.75n)$ goes to 0 exponentially fast

and is for large n much smaller than $\frac{1}{n^4}$.

3. The joint density of X and Y is

$$f(x, y) = \frac{1}{y} e^{-xy}$$

for $x > 0$ and $y > 1$, and 0 otherwise.
(a) Compute the conditional density of X, given $Y = y$. Do you recognize the distribution?

$$f_X(x \mid Y = y) = \frac{f(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_0^\infty \frac{1}{y} e^{-xy} \, dx = -\frac{1}{y^2} e^{-xy} \Big|_0^\infty = \frac{1}{y^2}$$

so

$$f_X(x \mid Y = y) = \frac{\frac{1}{y} e^{-xy}}{\frac{1}{y^2}} = \underline{\underline{y e^{-xy}}}$$

$(y \geq 1, \quad x \geq 0)$

(Exponential (y))

(b) Compute $E(X \mid Y = y)$.

$$E(X \mid Y = y) = \frac{1}{y}.$$

(Expectation of the exponential.)

4. Again, each minute, your score is independently chosen at random among the three numbers 1, 2, and 3; however, now you also (independently) roll a fair die each minute. You continue doing this until you roll a 6. Let N be the number of minutes the game lasts, and S the sum of all your scores. (Note that the die rolls do not contribute into your scores, but are used to decide when the game ends. For example, if your first three die rolls are 4, 1, 6, and your first three scores are 2, 1, 1 then $S = 2 + 1 + 1 = 4$.)
(a) Compute ES and $\text{Var}(S)$. (Help: variance of a Geometric(p) random variable is $\frac{1-p}{p^2}$.)

N is Geometric $\left(\frac{1}{6}\right)$, so that

$EN = 6$, $\quad \text{Var}(N) = \dfrac{5/6}{(1/6)^2} = 30$

$S = \sum_{i=1}^{N} X_i$, X_i i.i.d. $\quad EX_1 = \dfrac{1+2+3}{3} = 2$

$\text{Var}(X_1) = \dfrac{1^2+2^2+3^2}{3} - 4 = \dfrac{14}{3} - 4 = \dfrac{2}{3}$

So $\underline{\underline{ES}} = 6 \cdot 2 = \underline{\underline{12}}$ and

$\underline{\underline{\text{Var}(S)}} = \text{Var}(X_1)\, EN + (EX_1)^2\, \text{Var}(N)$

$= \frac{2}{3} \cdot 6 + 4 \cdot 30 = \underline{\underline{124}}$

(b) Compute $E[N \cdot S]$.

$$E[NS] = \sum_{n=1}^{\infty} E[NS \mid N=n]\, P(N=n)$$

$$= \sum_{n=1}^{\infty} n \cdot E[S_n] \cdot P(N=n)$$

$$= \sum_{n=1}^{\infty} n^2 \cdot 2\, P(N=n) = 2E(N^2)$$

$$= 2(\text{Var}(N) + (EN)^2) = 2(30+36) = 2 \cdot 66$$

$$= \underline{\underline{132}}$$