Math 135B, Spring 2011.
Apr. 20, 2011.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Calculators, books or notes are not allowed. Unless you are directed to do so, or it is required for further work, do *not* evaluate complicated expressions to give the result as a decimal number.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. In a group of $3n$ people, n people wear a red hat, n people wear a green hat, and n people wear a blue hat. The are seated at random on a row of $3n$ chairs; n of the chairs are red, n are green, n are blue. Let N_n be the number of people that sit in a chair of the same color as their hat.

(a) Compute EN_n.

$$N_n = \sum_{i=1}^{3n} \underbrace{I\{\text{person } i \text{ has the same color hat and chair}\}}_{I_i}$$

$$EI_i = \frac{1}{3}$$

$$EN_n = \frac{1}{3} \cdot 3n = n$$

(b) Compute $\mathrm{Var}(N_n)$.

$$EI_iI_j = \begin{cases} \frac{1}{3} \cdot \frac{n-1}{3n-1} & \text{if } i \text{ and } j \text{ hats are of the same color} \\ \frac{1}{3} \cdot \frac{n}{3n-1} & \text{if } i \text{ and } j \text{ hats are of different color} \end{cases}$$

$$EN_n^2 = \sum EI_i + \sum_{\substack{j \neq i \\ i,j \text{ same color hat}}} EI_iI_j + \sum_{\substack{i,j \text{ diff.} \\ \text{color hat}}} EI_iI_j$$

$$= n + \frac{1}{3} \frac{n-1}{3n-1} \cdot 3n \cdot (n-1) + \frac{1}{3} \frac{n}{3n-1} \cdot 3n \cdot 2n$$

$$\mathrm{Var}(N_n) = n + \frac{n(n-1)^2}{3n-1} + \frac{2n^3}{3n-1} - n^2$$

$$= \frac{2n^2}{3n-1}$$

(c) Determine a number c so that $\frac{1}{n}N_n \to c$, in probability, as $n \to \infty$.

$$Y_n = \frac{1}{n} N_n \qquad EY_n = 1$$

$$\mathrm{Var}(Y_n) = \frac{1}{n^2} \mathrm{Var}(N_n)$$

$$= \frac{2}{3n-1}$$

$$\xrightarrow[n \to \infty]{} 0$$

So, $Y_n \to 1$ in probability

2. The joint density of X and Y is given by

$$f(x,y) = \frac{e^{-3y}}{y}, \qquad \text{for } 0 < x < 3y,$$

and 0 otherwise. Compute the conditional density of X given $Y = y$. Do you recognize this distribution? Compute also $E(X|Y = y)$.

$$f_Y(y) = 3y \cdot \frac{e^{-3y}}{y} = 3e^{-3y} \qquad (y > 0)$$

$$f_X(x \mid Y = y) = \underline{\underline{\frac{1}{3y}}} \qquad 0 < x < 3y$$

(Uniform on $[0, 3y]$.)

$$E[X \mid Y = y] = \frac{1}{2} \cdot 3y = \underline{\underline{\frac{3}{2}y}} \, .$$

3. A casino offers the following game. Roll a fair die. If you roll 1 or 2, you lose \$2 (i.e., you win −\$2); if you roll 3, 4, or 5, you win or lose nothing; and if you roll 6, you win \$1.

(a) Assume you play once. Compute the moment generating function of the amount you win.

$X =$ amount you win $\quad P(X=-2) = \frac{1}{3}$

$P(X=0) = \frac{1}{2} \qquad EX = -\frac{2}{3} + \frac{1}{6} = \underline{\underline{-\frac{1}{2}}}$

$P(X=1) = \frac{1}{6}$

$$\varphi(t) = \frac{1}{3}e^{-2t} + \frac{1}{2} + \frac{1}{6}e^{t}$$

(b) Assume you play the game n times (independently). Compute the moment generating function of the combined amount S_n you win.

By independence

$$\varphi_{S_n}(t) = \left(\frac{1}{3}e^{-2t} + \frac{1}{2} + \frac{1}{6}e^{t}\right)^n$$

(c) Assume you play the game n times. Let p_n be the probability that your losses do not exceed $\frac{n}{4}$ dollars (i.e., you win at least $-\frac{n}{4}$ dollars). Explain how you would find a good upper bound for p_n, but do not carry out the computation. Is this probability smaller or larger than $\frac{1}{n^3}$ for large n?

$a = -\frac{1}{4} > EX$, so

$$I(a) = \max_{t>0}\left[-\frac{1}{4}t + \log\left(\frac{1}{3}e^{-2t} + \frac{1}{2} + \frac{1}{6}e^{t}\right)\right]$$

> 0.

$$p_n = P\left(S_n \geq -\frac{1}{4}n\right) \leq e^{-I(a)n},$$

much smaller than $\frac{1}{n^3}$.

4. Start with an empty pot. Each round, 3 people each toss a fair coin. If they all toss Tails, the game is over. Otherwise, each person who tosses Heads adds \$1 into the pot, and the game continues with the next round. Let N be the number of rounds on which a nonzero amount is added to the pot. Let S be the dollar amount in the pot after the game is over. (Help: variance of a Geometric(p) random variable is $\frac{1-p}{p^2}$.)

(a) Determine the expectation and variance of N.

N is Geometric$\left(\frac{1}{8}\right) - 1$, so $\underline{EN = 7}$

$$\underline{\mathrm{Var}(N)} = \frac{1-\frac{1}{8}}{\left(\frac{1}{8}\right)^2} = \underline{56}$$

(b) Determine the conditional distribution of S given $N = 2$.

The distribution of one contribution to the pot, conditioned on being nonzero:

$$X_1 = \begin{cases} 1 & \text{w.p. } 3/7 \\ 2 & \text{w.p. } 3/7 \\ 3 & \text{w.p. } 1/7 \end{cases}$$

Now, X_1 and X_2 i.i.d. with the above distr.,

$P(S=k \mid N=2) = P(X_1+X_2=k)$, $k=2,3,4,5,6$

Answer

k	$P(S=k \mid N=2)$
2	$(3\cdot3)/49 = 9/49$
3	$(3\cdot3+3\cdot3)/49 = 18/49$
4	$(3\cdot1+3\cdot3+1\cdot3)/49 = 15/49$
5	$(3\cdot1+1\cdot3)/49 = 6/49$
6	$(1\cdot1)/49 = 1/49$

(c) Determine the expectation and variance of S.

We have $S = \sum_{i=1}^{N} X_i$, where N is as in (a)

and $X_1, X_2, \ldots$ are i.i.d., distr. as X_1 in (b).

$$EX_1 = \frac{1}{7}(1\cdot3 + 2\cdot3 + 3\cdot1) = \frac{12}{7}$$

$$\mathrm{Var}(X_1) = \frac{1\cdot3 + 4\cdot3 + 9\cdot1}{7} - \left(\frac{12}{7}\right)^2 = \frac{24}{7} - \left(\frac{12}{7}\right)^2$$

$$= \frac{7\cdot24 - 12^2}{49} = \frac{168-144}{49} = \frac{24}{49}$$

$$\underline{\mathrm{Var}(S) = 7\cdot\frac{24}{49} + \left(\frac{12}{7}\right)^2\cdot 56}$$