Math 135B, Spring 2011.
May. 18, 2011.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____________________

ID#: ____________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Calculators, books or notes are not allowed. Unless you are directed to do so, or it is required for further work, do *not* evaluate complicated expressions to give the result as a decimal number.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. You have three coins: A (Heads probability 0.2), B (Heads probability 0.4), and C (Heads probability 0.6). Your plan is to toss one of the three coins each minute. Start by tossing coin A. Subsequently, if you toss Heads, you toss coin A next minute. If you toss Tails, you choose at random (with equal probability) coin B or coin C for your next toss.

(a) Determine the transition matrix of the Markov chain that keeps track of the coin you toss each minute.

$$P = \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.6 & 0.2 & 0.2 \end{bmatrix}$$

(b) Determine the (long-term) proportion of minutes you toss coin A; do the same for coins B and C.

$$[\pi_1, \pi_2, \pi_3]\, P = [\pi_1, \pi_2, \pi_2]$$

$$0.2\,\pi_1 + 0.4\,\pi_2 + 0.6\,\pi_3 = \pi_1$$
$$0.4\,\pi_1 + 0.3\,\pi_2 + 0.2\,\pi_3 = \pi_2$$
$$\pi_1 + \pi_2 + \pi_3 = 1$$
$$-0.8\,\pi_1 + 0.4\,\pi_2 + 0.6(1-\pi_1-\pi_2) = 0$$
$$0.4\,\pi_1 - 0.7\,\pi_2 + 0.2(1-\pi_1-\pi_2) = 0$$
$$1.4\,\pi_1 + 0.2\,\pi_2 = 0.6$$
$$-0.2\,\pi_1 + 0.9\,\pi_2 = 0.2 \qquad 6.5\,\pi_2 = 2$$

$$\boxed{\pi_2 = \frac{4}{13}}, \quad \boxed{\pi_1 = \frac{5}{13}}, \quad \boxed{\pi_3 = \frac{4}{13}}$$

(You may use $\pi_2 = \pi_3$, by symmetry, although not used here.)

Answer:

coin	long-term prop
A	5/13
B	4/13
C	4/13

(c) Determine the asymptotic proportion of minutes you toss Heads.

$\frac{5}{15}$ (the same as for coin A)

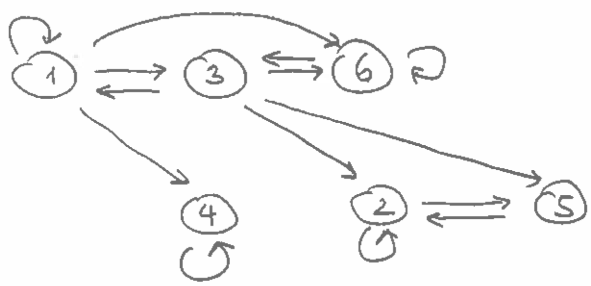
(c) Explain how you would compute the probability that you toss Heads on tosses 3 and 10. Do not carry out the computation. Assuming the starting toss is toss 0:

this is $P($coin A on toss 4, coin A on toss 11$)$

$$= P_{11}^{4}\, P_{11}^{7} \quad (\text{which is } \underline{\text{not}}\ P_{11}^{11}\,!)$$

2. Determine the transient and recurrent classes of the Markov chain with the following transition matrix:

$$
\begin{array}{c c}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} &
\begin{bmatrix}
0.1 & 0 & 0.1 & 0.1 & 0 & 0.7 \\
0 & 0.1 & 0 & 0 & 0.9 & 0 \\
0.1 & 0.1 & 0 & 0 & 0.1 & 0.7 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0.1 & 0 & 0.9 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$



$\{1, 3, 6\}$ transient (as not closed)

$\{4\}$ recurrent (absorbing)

$\{2, 5\}$ recurrent (as closed)

3. In a branching process, every individual has 0 descendants with probability $\frac{2}{5}$, and 1, 2, or 3 descendants with probability $\frac{1}{5}$ each. Start the process at generation 0 with a single ancestor, and let X_n be the population size in generation n.

(a) Compute the probability that the branching process ever dies out. (Help: $s^3 + s^2 - 4s + 2 = (s-1)(s^2 + 2s - 2)$).

$$E X_1 = \frac{1+2+3}{5} = \frac{6}{5} = \mu > 1,$$

$$\text{m.g.f.}: \varphi(s) = \frac{2}{5} + \frac{s + s^2 + s^3}{5}$$

$$\varphi(s) = s \qquad 2 + s + s^2 + s^3 = 5s$$

$$s^3 + s^2 - 4s + 2 = 0$$

$$(s-1)(s^2 + 2s - 2) = 0$$

$$s = \frac{-2 + \sqrt{12}}{2} = \sqrt{3} - 1$$

Answer: $\pi_0 = \sqrt{3} - 1$.

(b) Compute $E(X_7 | X_3 = 4)$.

$$4\mu^4 = 4\left(\frac{6}{5}\right)^4$$

4. A Markov chain has six states 1, 2, 3, 4, 5, 6. The following transitions happen with probability $p \in (0,1)$: $1 \to 2, 2 \to 3, 3 \to 1, 4 \to 5, 5 \to 6, 6 \to 4$; the following transitions happen with probability $1-p$: $1 \to 4$, $4 \to 1$, $2 \to 5$, $5 \to 2$, $3 \to 6$, $6 \to 3$; and there are no other possible transitions.

(a) Write down the transition matrix P and determine the invariant distribution for this chain. (No long calculations here!)

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} & p & & 1-p & & \\ & & p & & 1-p & \\ p & & & & & 1-p \\ 1-p & & & & p & \\ & 1-p & & & & p \\ & & 1-p & p & & \end{bmatrix} \end{array}$$

Doubly stochastic:

$$\pi = \left[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right]$$

(b) For all states i and j, determine the limit, as $n \to \infty$, of P_{ij}^n. Carefully verify the conditions of any theorems you use.

The limit is $\frac{1}{6}$, indep. of i and j, as the chain is irreducible and aperiodic (can get from 1 ~~to~~ 1 in 3 steps $1 \to 2 \to 3 \to 1$ and in 2 steps $1 \to 4 \to 1$).

(c) How would the answer in (b) change if the possible transition from 1 are $1 \to 1$ (prob. $\frac{1}{2}$), $1 \to 2$ (prob. $\frac{p}{2}$), and $1 \to 4$ (prob. $\frac{1-p}{2}$), while all the other transition probabilities remain the same?

Now you spend a Geometric$(\frac{1}{2})$ time at 1 for each visit. So the time spent at 1 is twice the time spent at other states and the inv. distr. is $\left[\frac{2}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right]$

$$\lim_{n\to\infty} P_{ij}^n = \begin{cases} 2/7 & \text{if } j=1 \\ 1/7 & \text{otherwise} \end{cases} \quad \text{(indep. of } i\text{)}$$