A collection of homework problems

Homework 1

1. Fifteen married couples are at a dance lesson. Assume that husband-wife dance pairs are assigned at random. (a) What is the number of possible assignments? (b) What is the probability that each husband ends up dancing with his wife?

2. Three Finns and three Danes sit in a row at random. Compute: (a) the probability that the three Finns sit in three adjacent seats, and the same is true for the Danes; (b) the probability that the three Finns sit in three adjacent seats; (c) no two adjacent seats are occupied by citizens of the same nation.

3. Fifteen married couples are at a dance lesson, but now only five husbands and five wives are selected at random, then randomly paired. (a) What is the number of possible dancing arrangements? (b) What is the probability that John Smith, one of the husbands, dances with Jane Smith, his wife?

4. You are walking on points in the plane with integer coordinates. Each time you can move either one unit up or one unit right; for example, from $(2,4)$ you can move either to $(3,4)$ or $(2,5)$. You start at the origin $(0,0)$ want to reach $(4,3)$. (a) How many possible routes do you have? (b) You choose one of the routes at random. What is the probability that you visit $(2,2)$ on your route?

5. Twelve people are divided at random into three committees: A (3 people) B (4 people) and C (5 people). What is the probability that A consists of the 3 youngest people and C of the 5 oldest people?

Homework 2

1. Three tours, A, B, and C, are offered to a group of 100 tourists. It turns out that 28 tourists sign for A, 26 for B, 16 for C, 12 for both A and B, 4 for both A and C, 6 for both B and C, and 2 for all three tours. (a) What is the probability that a randomly chosen tourist is taking none of these tours? (b) What is the probability that a randomly chosen tourist is taking exactly one of these tours? (c) What is the probability that two randomly chosen tourists are both taking at least one of these tours?

2. The *Poker Dice* is a game in which a player rolls 5 dice, and the hand is observed as in poker, except there are no flushes (because there are no suits), 1 may only be the lowest number in a straight, and five of a kind is an additional possibility. Compute the probabilities of all seven hands: one pair, two pairs, three of a kind, straight, full house, four of a kind, and five of a kind.

3. An urn contains 5 red, 6 green, and 8 blue balls. Take three balls out at random one by one (a) without and (b) with replacement. In each case compute the probability that the balls are of (1) the same color and (2) three different colors.

4. Assume a birthday is equally likely to be in any month of the year. In a group of 20 people, what is the probability that 4 months contain *exactly* 2 birthdays each and 4 months contain *exactly* 3 birthdays each?

5. You are dealt 13 cards from a shuffled deck of 52 cards. Compute the probability that (a) your hand lacks at least one suit, (b) you get the both Ace and King of at least one suit, (c) you get all four cards of at least one denomination (all Aces, or all Kings, or all Queens, ..., or all Twos).

6. Three boxes clearly labeled A, B, C contain three cards each. Here are the (known) values of the cards in each box: A: 1, 6, 8; B: 3, 5, 7; C: 2, 4, 9. Going first, Player 1 is free to select any of the three boxes, then chooses *at random* one of the cards from the box. Then Player 2 selects one of the remaining two boxes, and chooses *at random* a card from the box. The highest card wins. When you play the game, which of the two players would you prefer to be?

Homework 3

1. There are three bags: A (contains 2 white and 4 red balls), B (8 white, 4 red) and C (1 white 3 red). You select one ball at random from each bag, observe that exactly two are white, but forget which ball came from which bag. What is the probability that you selected a white ball from bag A?

2. Pick three cards one by one at random from a full deck of 52 cards (a) without replacement and (b) with replacement. Compute the probability that the first card is a hearts ($\heartsuit$) card, the second card is an Ace, and the third card is a hearts card.

3. A bag initially contains 5 white and seven black balls. Each time you select a ball at random and return it to the bag together with two balls of the same color. Compute the probability that (a) first two selected balls are black and the next two are white and (b) that among the first four selected balls, exactly two are black.

4. There are two bags: A (contains 2 white and 4 red balls), B (1 white, 1 red). Select a ball at random from A, then put it into B. Then select a final ball at random from B. Compute (a) the probability that the final ball is white and (b) the probability that the transferred ball is white given that the final ball is white.

5. Shuffle a full deck of cards and turn them over one by one until an Ace appears. It turns out that this happened exactly when the 20th card was turned over. What is the probability that the next card is (a) Ace of spades, (b) two of clubs?

6. Shuffle a full deck of cards and divide it into two stacks of 26 cards. A card is taken from the top of the first stack, and, after its value is observed, put into the second stack. The second stack is then reshuffled, a card is dealt from the top, and its value observed. What is the probability that the two values are the same?

7. A bag contains 8 black and 4 white balls. Three players, A, B, and C, select a ball at random in this order A, B, C, A, B, C, etc., until a white ball is selected, and the player who does that is the winner. Compute the winning probabilities for the three players if the balls are (a) replaced after each selection and (b) are never replaced.

Homework 4

1. A bag contains 8 white, 4 black and 2 grey balls. You choose two balls at random from a bag, without replacement, and win \$2 for each black ball but lose \$1 for each white ball. Let X denote your winnings. Compute the p.m.f. of X. Compute EX.

2. Let X be the difference between the number of Heads and the number of Tails in n tosses of a fair coin. Compute the p.m.f. of X.

3. A group of 100 people is tested for a disease, for which there is an infallible but expensive blood test. They are divided into 10 groups of 10 people each, then the people in each group pool their blood. If a test for a group comes out negative, everybody in that group is healthy and no more tests are done. Otherwise, the blood of each of the 10 people in the group is tested separately. Assume that the probability that a person has the disease is, independently, 0.1. Compute the expected number of tests performed.

4. A bag contains 5 red and 5 blue balls. Select two at random without replacement. If they are of the same color you win \$1.10 otherwise you lose \$1. Let X be your winnings. Compute EX and $\mathrm{Var}(X)$.

5. A multiple choice exam has 5 questions, with three choices for each question. The passing score is at least four correct answers. (a) What is the probability that a student who answers each question at random passes the test? (b) Assuming a class has 50 students, and all answer each question at random, what is the expected number of students that pass the test?

6. Assume the suicide rate is 1 per 100,000 people per month, and a city has 400,000 inhabitants. (a) Find the probability that there will be 8 or more suicides next month in this city. (b) Find the probability that next year there will be at least two months with 8 or more suicides. (c) Counting the next month as month 1, what is the probability that the first month to have 8 or more suicides will be month i?

Homework 5

1. Assume that weekly sales of diesel fuel at a gas station are X tons, where X is a random variable with density function

$$f(x) = \begin{cases} c(1-x)^4 & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute c. (b) Compute EX. (c) The current tank (of capacity 1) will be emptied and out for repairs next week; to save money, the station wants to rent a tank for the week with capacity just large enough that its supply will be exhausted with probability 0.01. What is the capacity of the tank the station needs to rent?

2. Alice arrives at a train station at a time uniformly distributed between 8 p.m. and 9 p.m., and boards the first train that arrives. There are two types of trains that arrive between 7 a.m. and midnight: those that head to town A arrive every 15 minutes starting at 7 a.m., and those that head to town B arrive every 15 minutes starting at 7:05 a.m.. (a) How long does Alice wait, on the average? (b) What proportion of days does she go to A?

3. If X is an Exponential(1) random variable, compute the density of (a) $Y = \log X$ and (b) $Z = (\log X)^2$.

4. Assume A is uniform in $[0, 5]$. Compute the probability that the equation $4x^2 + 4Ax + A + 2 = 0$ has two real roots.

5. January snowfall, in inches, in Truckee, California has expectation 50 and standard deviation 35. (These are close to true numbers, from the ranger station website.) Assume normal distribution and year to year independence. What is the probability that, starting from the next January, it will take at least 11 years to get January snowfall over 100 inches?

6. A fair die is rolled 1000 times. Let A be the event that the number of 6's is in the interval $[150, 200]$, and B the event that the number of 5's is exactly 200. (a) Approximate $P(A)$. (b) Approximate $P(A|B)$.

Homework 6

1. A fair die is rolled five times. (a) Compute the joint p.m.f. of X and Y, where X is the first number rolled and Y is the largest of the five numbers rolled. (b) Compute the joint p.m.f. of X and Y, where X is the number of 1's rolled and Y is the largest of the five numbers rolled. (c) Compute the joint p.m.f. of X and Y, where X is the first number rolled and Y is the number of 1's rolled.

2. Joint density of (X, Y) is given by

$$f(x,y) = c(x^2 + \frac{xy}{2}), \quad 0 < x < 1, \, 0 < y < 2.$$

(a) Compute c. (b) Compute the density of X. (c) Compute $P(X > Y)$. (d) Compute the conditional probability $P(Y > 1/2|X < 1/2)$. (e) Find EY.

3. Mr. Smith arrives at a location at a time uniformly distributed between 12:15 and 12:45, while Mrs. Smith independently arrives at the same location at a time uniformly distributed between 12 and 1 (all times p.m.). (a) Compute the probability that the first person to arrive waits no longer than 5 minutes. (b) Compute the probability that Mr. Smith arrives first.

4. Joint density of (X, Y) is given by

$$f_1(x,y) = \begin{cases} x \cdot e^{-(x+y)} & x, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Are X and Y independent? (b) Repeat with joint density

$$f_2(x,y) = \begin{cases} 2 & 0 < x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Homework 7

1. Joint density of (X, Y) is given by

$$f(x, y) = xe^{-x(y+1)}, \quad x, y > 0.$$

(a) Find the conditional density of Y given $X = x$. (b) Compute the density of $Z = XY$.

2. Assume that $X_1, \ldots, X_{10}$ are independent Exponential(λ) random variables. Compute the densities of $H = \max\{X_1, \ldots, X_{10}\}$ and $L = \min\{X_1, \ldots, X_{10}\}$.

3. Select a point (X, Y) at random from the square $[-1, 1] \times [-1, 1]$. Compute (a) $E(|X| + |Y|)$, (b) $E|XY|$, and (c) $E|X - Y|$.

4. Suppose that Alice and Bob, each randomly and independently, choose 3 out of 10 objects without replacement. Find the expected number of objects that are (a) chosen by both Alice and Bob, (b) not chosen by either Alice or Bob, (c) chosen by exactly one of them.

5. A coin has probability p of Heads. Toss this coin n times, and let X be the number of tosses, from toss 2 on, that have different outcome than the previous toss. Compute EX.

6. Let (X, Y) be a random point of in the square $\{(x, y) : 0 \le x, y \le 1\}$. (a) Compute the density of $Z = XY$, EZ, and $\mathrm{Var}(Z)$. (b) Assume that 1000 such points (X_i, Y_i), $i = 1, \ldots, 1000$, are chosen independently, and approximate the probability $P(X_1Y_1 + X_2Y_2 + \cdots + X_{1000}Y_{1000} < 255)$.