Interlude: Practice Midterm 1

This practice exam covers the material from the chapters 9 through 11. Give yourself 50 minutes to solve the four problems, which you may assume have equal point score.

1. Recall that a full deck of cards contains 52 cards, 13 cards of each of the four suits. Distribute card at random to 13 players, so that each gets 4 cards. Let $N_1$ be the number of players whose four cards are of the same suit and $N_2$ the number of players whose four cards are of four different suits. Using the indicator trick, compute $E(N_1)$ and $E(N_2)$. Which one is larger?

2. Consider the following game, which will also appear in problem 4. Toss a coin with probability $p$ of Heads. If you toss Heads, you win $2, if you toss Tails, you win $1.
   (a) Assume that you play this game three times and let $S_3$ be your combined winnings. Compute the moment generating function of $S_3$, that is, $E(e^{tS_3})$. (You do not need to multiply out.)
   (b) Now you roll a fair die and you play the game as many times as the number you roll. Let $Y$ be your total winnings. Compute $E(Y)$ and $\text{Var}(Y)$.

3. The joint density of $X$ and $Y$ is
   \[ f(x, y) = \frac{e^{-x/y}e^{-y}}{y} \]
   for $x > 0$ and $y > 0$, and 0 otherwise. Compute $E(X|Y = y)$.

4. Again, consider the following game. Toss a coin with probability $p$ of Heads. If you toss Heads, you win $2, if you toss Tails, you win $1. Assume that you start with no money and you have to quit the game when your winnings match or exceed the dollar amount $n$. (For example, assume $n = 5$ and you have $3$: if your next toss is Heads, you collect $5 and quit; if your next toss is Tails, you play once more. Note that, at the amount you quit, your winnings will be either $n$ or $n + 1.$) Let $p_n$ be the probability that you will quit with winnings exactly $n$.
   (a) What is $p_1$? What is $p_2$?
   (b) Write down the recursive equation which expresses $p_n$ with $p_{n-1}$ and $p_{n-2}$.
   (c) Solve the recursion.
Solutions to Practice Midterm 2

1. Recall that a full deck of cards contains 52 cards, 13 cards of each of the four suits. Distribute cards at random to 13 players, so that each gets 4 cards. Let \( N_1 \) be the number of players whose four cards are of the same suit and \( N_2 \) the number of players whose four cards are of four different suits. Using the indicator trick, compute \( E(N_1) \) and \( E(N_2) \). Which one is larger?

Solution:
For \( N_1 \), let 
\[
I_i = I_{\text{player } i \text{ has four cards of one suit}},
\]
and then \( N_1 = I_1 + \ldots + I_{13} \). Observe that:

- number of ways to pick 4 cards of 52 card deck is \( \binom{52}{4} \);
- number of choices of a suit is 4; and
- after picking a suit, number of ways to pick 4 cards of that suit is \( \binom{13}{4} \).

Therefore
\[
EI_i = \frac{4\binom{13}{4}}{\binom{52}{4}},
\]
and finally,
\[
EN_1 = \frac{4\binom{13}{4}}{\binom{52}{4}} \cdot 13.
\]

For \( N_2 \), let 
\[
I'_i = I_{\text{player } i \text{ has four different suits}},
\]
and then \( N_2 = I'_1 + \ldots + I'_{13} \). This time
\[
EI'_i = \frac{13^4}{\binom{52}{4}},
\]
and so
\[
EN_2 = 13 \cdot \frac{13^4}{\binom{52}{4}}.
\]
Clearly \( EN_2 > EN_1 \).

2. Consider the following game, which will also appear in problem 4. Toss a coin with probability \( p \) of Heads. If you toss Heads, you win $2, if you toss Tails, you win $1.

(a) Assume that you play this game three times and let \( S_3 \) be your combined winnings. Compute the moment generating function of \( S_3 \), that is, \( E(e^{tS_3}) \). (You do not need to multiply out.)
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Solution:

\[ E(e^{tS_1}) = (e^{2t} \cdot p + e^t \cdot (1 - p))^3. \]

(b) Now you roll a fair die and you play the game as many times as the number you roll. Let \( Y \) be your total winnings. Compute \( E(Y) \) and \( \text{Var}(Y) \)

Solution:

Let \( Y = X_1 + \ldots + X_N \), where \( X_i \) are independently and identically distributed with \( P(X_1 = 2) = p \) and \( P(X_1 = 1) = 1 - p \), and \( P(N = k) = \frac{1}{6} \), for \( k = 1, \ldots, 6 \). Then we know that

\[
EY = EN \cdot EX_1, \quad \text{Var}(Y) = \text{Var}(X_1) \cdot EN + (EX_1)^2 \cdot \text{Var}(N).
\]

We have

\[ EN = \frac{7}{2}, \quad \text{Var}(N) = \frac{35}{12}. \]

Moreover,

\[ EX_1 = 2p + 1 - p = 1 + p, \]

and

\[ EX_1^2 = 4p + 1 - p = 1 + 3p, \]

so that

\[ \text{Var}(X_1) = 1 + 3p - (1 + p)^2 = p - p^2. \]

The answer is

\[
EY = \frac{7}{2} \cdot (1 + p), \quad \text{Var}(Y) = (p - p^2)\frac{7}{2} + (1 + p)^2\frac{35}{12}.
\]

3. The joint density of \( X \) and \( Y \) is

\[ f(x, y) = \frac{e^{-x/y}e^{-y}}{y} \]

for \( x > 0 \) and \( y > 0 \), and 0 otherwise. Compute \( E(X|Y = y) \)

Solution:

We have

\[ E(X|Y = y) = \int_0^\infty x f_X(x|Y = y) \, dx. \]
As
\[
f_Y(y) = \int_0^\infty \frac{e^{-x/y}e^{-y}}{y} \, dx
= \frac{e^{-y}}{y} \int_0^\infty e^{-x/y} \, dx
= \frac{e^{-y}}{y} \left[ ye^{-x/y} \right]_{x=0}^{x=\infty}
= e^{-y}, \text{ for } y > 0,
\]
we have
\[
f_X(x|Y = y) = \frac{f(x, y)}{f_Y(y)}
= \frac{e^{-x/y}}{y}, \text{ for } x, y > 0,
\]
and so
\[
E(X|Y = y) = \int_0^\infty \frac{x}{y} e^{-x/y} \, dx
= y \int_0^\infty ze^{-z} \, dz
= y.
\]

4. Again, consider the following game. Toss a coin with probability \( p \) of Heads. If you toss Heads, you win $2, if you toss Tails, you win $1. Assume that you start with no money and you have to quit the game when your winnings match or exceed the dollar amount \( n \). (For example, assume \( n = 5 \) and you have $3: if your next toss is Heads, you collect $5 and quit; if your next toss is Tails, you play once more. Note that, at the amount you quit, your winnings will be either \( n \) or \( n + 1 \).) Let \( p_n \) be the probability that you will quit with winnings exactly \( n \).

(a) What is \( p_1 \)? What is \( p_2 \)?

Solution:
We have
\[
p_1 = 1 - p
\]
and
\[
p_2 = (1 - p)^2 + p.
\]
Also, \( p_0 = 1. \)
(b) Write down the recursive equation which expresses $p_n$ with $p_{n-1}$ and $p_{n-2}$.

Solution:
We have

$$ p_n = p \cdot p_{n-2} + (1 - p)p_{n-1}. $$

(c) Solve the recursion.

Solution:
We can use

$$ p_n - p_{n-1} = (-p)(p_{n-1} - p_{n-2}) = (-p)^{n-1}(p_1 - p_0). $$

Another possibility is to use the characteristic equation $\lambda^2 - (1 - p)\lambda - p = 0$ to get

$$ \lambda = \frac{1 - p \pm \sqrt{(1 - p)^2 + 4p}}{2} = \frac{1 - p \pm (1 + p)}{2} = \begin{cases} 1 \\ -p \end{cases}. $$

This gives

$$ p_n = a + b(-p)^n, $$

with

$$ a + b = 1, \ a - bp = 1 - p. $$

We get

$$ a = \frac{1}{1 + p}, \ b = \frac{p}{1 + p}, $$

and then

$$ p_n = \frac{1}{1 + p} + \frac{p}{1 + p}(-p)^n. $$