Interlude: Practice Midterm 2

This practice exam covers the material from the chapters 12 through 15. Give yourself 50 minutes to solve the four problems, which you may assume have equal point score.

1. Suppose that whether it rains any day or not depends on whether it did so through the previous two days. If it rained yesterday and today, then it will rain tomorrow with probability 0.7; if it rained yesterday but not today, it will rain tomorrow with probability 0.4; if it did not rain yesterday but rained today, it will rain tomorrow with probability 0.5; if it did not rain yesterday and also did not today, it will rain tomorrow with probability 0.2.

(a) Let $X_n$ be a Markov chain with 4 states, \((R, R), (N, R), (R, N), (N, N)\), which code (weather yesterday, weather today) with $R =$ rain and $N =$ no rain. Write down the transition probability matrix for this chain. (b) Today is Wednesday and it is raining. It has also rained yesterday. Explain how you would compute the probability that it will rain on Saturday. Do not carry out the computation.

(c) Under the same assumption as in (b), explain how you would approximate the probability of rain on a day exactly a year from today. Carefully justify your answer, but do not carry out the computation.

2. Consider the Markov chain with states 1, 2, 3, 4, 5, given by the following transition matrix:

\[
P = \begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

Specify all the classes and determine whether they are transient or recurrent.

3. In a branching process, the number of descendants is determined as follows. An individual first tosses a coin that comes out heads with probability $p$. If this coin comes out tails, the individual has no descendants. If this coin comes heads, the individual has 1 or 2 descendants, each with probability $\frac{1}{2}$.

(a) Compute $\pi_0$, the probability that the branching process dies out eventually. Your answer will course depend on the parameter $p$.

(b) Write down the expression for the probability that the branching process is still alive at generation 3. Do not simplify.

4. A random walker is in one of the four states, 0, 1, 2, or 3. If she at $i$ at some time, she makes the following transition. With probability $\frac{1}{2}$ she moves from $i$ to $(i + 1) \mod 4$ (that is, if she is at 0 she moves to 1, from 1 she moves to 2, from 2 to 3, and from 3 to 0). With probability $\frac{1}{2}$, she moves to a random state among the four states, each chosen with equal probability.

(a) Show that this chain has invariant distribution and compute it (Take a good look at the transition matrix before you start solving this).
(b) After the walker makes many steps, compute the proportion of time she spends at 1. Does the answer depend on the chain’s starting point?

(c) After the walker makes many steps, compute the proportion of time she is at the same state as at the previous time.
Solutions to Practice Midterm 2

1. Suppose that whether it rains any day or not depends on whether it did so through the previous two days. If it rained yesterday and today, then it will rain tomorrow with probability 0.7; if it rained yesterday but not today, it will rain tomorrow with probability 0.4; if it did not rain yesterday but rained today, it will rain tomorrow with probability 0.5; if it did not rain yesterday and also did not today, it will rain tomorrow with probability 0.2.

(a) Let $X_n$ be a Markov chain with 4 states, $(R,R)$, $(N,R)$, $(R,N)$, $(N,N)$, which code (weather yesterday, weather today) with $R =$ rain and $N =$ no rain. Write down the transition probability matrix for this chain.

Solution:
Let $(R,R)$ be state 1, $(N,R)$ state 2, $(R,N)$ state 3, and $(N,N)$ state 4. The transition matrix is

$$P = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}.$$

(b) Today is Wednesday and it is raining. It has also rained yesterday. Explain how you would compute the probability that it will rain on Saturday. Do not carry out the computation.

Solution: If Wednesday is time 0, then Saturday is time 3. The initial state is given by the row $[1, 0, 0, 0]$, and it will rain on Saturday if we end up at state 1 or 2. Therefore, our solution is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \cdot P^3 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

that is, the sum of the first two entries of $[1, 0, 0, 0] \cdot P^3$.

(c) Under the same assumption as in (b), explain how you would approximate the probability of rain on a day exactly a year from today. Carefully justify your answer, but do not carry out the computation.

Solution:
The matrix $P$ is irreducible since the chain makes the following transitions with
positive probability: \((R, R) \rightarrow (R, N) \rightarrow (N, N) \rightarrow (N, R) \rightarrow (R, R)\). It is also aperiodic because the transition \((R, R) \rightarrow (R, R)\) has positive probability. Therefore, the probability can be approximated by \(\pi_1 + \pi_2\), where \([\pi_1, \pi_2, \pi_3, \pi_4]\) is the unique solution to \([\pi_1, \pi_2, \pi_3, \pi_4] \cdot P = [\pi_1, \pi_2, \pi_3, \pi_4]\) and \(\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1\).

2. Consider the Markov chain with states 1, 2, 3, 4, 5, given by the following transition matrix:

\[
P = \begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

Specify all the classes and determine whether they are transient or recurrent.

Solution:
Answer:
- \(\{2\}\) is transient;
- \(\{1, 3\}\) is recurrent;
- \(\{4, 5\}\) is recurrent.

3. In a branching process, the number of descendants is determined as follows. An individual first tosses a coin that comes out heads with probability \(p\). If this coin comes out tails, the individual has no descendants. If this coin comes heads, the individual has 1 or 2 descendants, each with probability \(\frac{1}{2}\).

(a) Compute \(\pi_0\), the probability that the branching process dies out eventually. Your answer will course depend on the parameter \(p\).

Solution:
The probability mass function for the number of descendants is

\[
\begin{pmatrix}
0 & 1 & \frac{1}{2} \\
1 - p & \frac{p}{2} & \frac{p}{2}
\end{pmatrix},
\]

and so

\[
E(\text{number of descendants}) = \frac{p}{2} + p = \frac{3p}{2}.
\]

If \(\frac{3p}{2} \leq 1\), i.e., \(p \leq \frac{2}{3}\), then \(\pi_0 = 1\). Otherwise, we need to compute \(\phi(s)\) and solve \(\phi(s) = s\). We have

\[
\phi(s) = 1 - p + \frac{p}{2}s + \frac{p}{2}s^2.
\]
and then

\[ s = 1 - p + \frac{p}{2}s + \frac{p}{2}s^2, \]

\[ 0 = ps^2 + (p - 2)s + 2(1 - p), \]

\[ 0 = (s - 1)(ps - 2(1 - p)) = 0. \]

We conclude that if \( p > \frac{2}{3} \), \( \pi_0 = \frac{2(1-p)}{p} \).

(b) Write down the expression for the probability that the branching process is still alive at generation 3. Do not simplify.

Solution:
The answer is \( 1 - \phi(\phi(\phi(0))) \) and we compute

\[ \phi(0) = 1 - p, \]

\[ \phi(\phi(0)) = 1 - p + \frac{p}{2}(1 - p) + \frac{p}{2}(1 - p)^2, \]

\[ 1 - \phi(\phi(\phi(0))) = 1 - \left(1 - p + \frac{p}{2}(1 - p) + \frac{p}{2}(1 - p)^2 + \frac{p}{2}(1 - p)^2 \right).\]

4. A random walker is in one of the four states, 0, 1, 2, or 3. If she at \( i \) at some time, she makes the following transition. With probability \( \frac{1}{2} \) she moves from \( i \) to \( (i + 1) \mod 4 \) (that is, if she is at 0 she moves to 1, from 1 she moves to 2, from 2 to 3, and from 3 to 0). With probability \( \frac{1}{2} \), she moves to a random state among the four states, each chosen with equal probability.

(a) Show that this chain has invariant distribution and compute it (Take a good look at the transition matrix before you start solving this).

Solution:
The transition matrix is

\[
\begin{bmatrix}
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}
\end{bmatrix}.
\]

As \( P \) is a doubly stochastic irreducible matrix, \( \pi = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right] \) is the unique invariant distribution. (Note that irreducibility is trivial, as all entries are positive).

(b) After the walker makes many steps, compute the proportion of time she spends at 1. Does the answer depend on the chain’s starting point?
Solution: The proportion equals to $\pi_1 = \frac{1}{4}$, independently of the starting point.

(c) After the walker makes many steps, compute the proportion of time she is at the same state as at the previous time.

Solution: The probability of staying at the same state is always $\frac{1}{8}$, which is the answer.