Erdős-Feller-Pollard renewal theorem

Example 1. Roll a fair die forever and let $S_m$ be the sum of outcomes of first $m$ rolls. Let $p_n = P(S_m$ ever equals $n)$. Estimate $p_{10,000}$.

One can write a linear recursion

$$
p_0 = 1,\]
$$

$$
p_n = \frac{1}{6} (p_{n-1} + \cdots + p_{n-6}),
$$

and then solve it, but this is a lot of work! Note that one should either modify the recursion for $n \leq 5$ or, much easier, define $p_n = 0$ for $n < 0$.

Example 2. Assume a random walk starts from 0 and jumps from $x$ either to $x+1$ or to $x+2$, with probability $p$ and $1-p$, respectively. What is now, approximately, the probability that the walk ever hits 10,000? The recursion is now much simpler

$$
p_0 = 1,\]
$$

$$
p_n = p \cdot p_{n-1} + (1-p) \cdot p_{n-2},
$$

and you should solve it, but again it would be nice if we can avoid the work.

Theorem. Assume that $f_1, \ldots, f_N \geq 0$ are given numbers, with $\sum_{k=1}^N f_k = 1$. Let $\mu = \sum_{k=1}^N k f_k$. Define the sequence $u_n$ as follows:

$$
\begin{align*}
u_n &= 0 \quad \text{if } n < 0, \\
u_0 &= 1, \\
u_n &= \sum_{k=1}^N f_k u_{n-k} \quad \text{if } n > 0.\end{align*}
$$

Assume that the greatest common divisor of the set $\{k : f_k > 0\}$ is 1. Then

$$
\lim_{n \to \infty} u_n = \frac{1}{\mu}.
$$

Examples. In Example 1, the answer is therefore $2/7$ and in the Example 1 it is $1/(2-p)$.

Proof. We can assume, without loss of generality that $f_N > 0$ (or else reduce $N$).
Define a Markov chain with state space \( S = \{0, 1, \ldots, N - 1\} \) by
\[
\begin{bmatrix}
    f_1 & 1 - f_1 & 0 & 0 & \ldots \\
    f_2 & 0 & 1 - f_1 - f_2 & 0 & \ldots \\
    \frac{f_3}{1 - f_1} & 0 & 0 & 1 - f_1 - f_2 - f_3 & \ldots \\
    \vdots & & & & \ddots & \vdots \\
    \frac{f_N}{1 - f_1 - \cdots - f_{N-1}} & 0 & 0 & 0 & \ldots 
\end{bmatrix}
\]
This is called renewal chain, it moves to the right (from \( x \) to \( x + 1 \)) on nonnegative integers, except for renewals, i.e., jumps to 0. At \( N - 1 \), the jump to 0 is certain (note that the matrix entry \( P_{N-1,0} \) is 1, since the sum of \( f_k \)'s is 1).

The chain is irreducible (you can get to \( N - 1 \) from anywhere, from there to 0, and from there anywhere) and aperiodic (if \( f_k > 0 \), you can get from 0 to 0 in \( k \) steps). Moreover, if \( X_0 = 0 \), then
\[
P(\text{first return time to 0 is } k)
\]
clearly equals \( f_1 \) if \( k = 1 \), then for \( k = 2 \) it equals
\[
(1 - f_1) \cdot \frac{f_2}{1 - f_1} = f_2,
\]
then for \( k = 3 \) it equals
\[
(1 - f_1) \cdot \frac{1 - f_1 - f_2}{1 - f_1} \cdot \frac{f_3}{1 - f_1 - f_2} = f_3,
\]
and so on. We conclude that (recall again that \( X_0 = 0 \))
\[
P(\text{first return time to 0 is } k) = f_k \quad \text{for all } k \geq 1.
\]
Then the mean return time to 0 is
\[
m_{00} = \sum_{k=1}^{N} k f_k = \mu.
\]
The next observation is that the probability \( P_{00}^n \) that the chain is at 0 in \( n \) steps is given by the recursion
\[
(1) \quad P_{00}^n = \sum_{k=1}^{n} P(\text{first return time to 0 is } k)P_{00}^{n-k},
\]
obtained simply by noting that you must return sometime by time $n$ in order to end up at 0: either you return for the first time at time $n$, or you return at some previous time $k$, and then you have to be back at 0 in $n-k$ steps.

The above formula (1) is always true (for every Markov chain). In this case, however, we note that for sure the first return time to 0 is at most $N$, so we can always sum to $N$ with the proviso that $P^{n-k}_0 = 0$ when $k > n$. So from (1) we get

\[(2) \quad P^n_{00} = \sum_{k=1}^{N} f_k P^{n-k}_{00}.
\]

The recursion for $P^n_{00}$ is the same as the recursion for $u_n$! The initial conditions are also the same, so $u_n = P^n_{00}$. It follows from the convergence theorem for Markov chains that

\[
\lim_{n \to \infty} u_n = \lim_{n \to \infty} P^n_{00} = \frac{1}{m_{00}} = \frac{1}{\mu},
\]

which ends the proof. □