1. Assume that you have \( n \) pairs of socks in a drawer, each of different color. Pick \( n \) socks from the drawer at random. Let \( X \) be the number of matching pairs you get. Use the indicator trick to compute \( E(X) \).

Extra credit. Assume \( n = 100 \). How many socks should pull out of the drawer at random so that you have a good (say at least 90%) chance of getting at least one matching pair. (You won’t receive any credit if your answer is larger than 50!)

2. Toss a fair coin until you get heads for the first time. After each toss which comes out tails, roll a fair die, and collect as many dollars as the number which shows up on the die. Let \( X \) be your expected winnings. Compute, by conditioning, \( E(X) \) and \( Var(X) \).

3. The joint density of \( X \) and \( Y \) is given by

\[
f(x, y) = \frac{e^{-y}}{y}, \quad 0 < x < y, 0 < y < \infty.
\]

Compute the conditional expectations \( E(X|Y = y) \) and \( E(X^2|Y = y) \).

4. Three white and three black balls are distributed in two urns in such a way that each contains three balls. At each step, draw one ball from each urn and place the ball drawn from the first urn into the second urn, and, conversely, the ball drawn from the second urn into the first urn. Let \( X_n \) be the number of white balls in the first urn.

(a) Write down the transition matrix for the Markov chain \( X_n \).

(b) Assume that, initially, you choose 3 of the balls at random, put them into urn 1, and put the remaining 3 into urn 2. Then perform 2 of the steps described above. Write down the probability mass function of \( X_2 \).

**PRACTICE EXAM 2**

1. Roll a fair die once, let \( N \) be the number you roll. Then choose independent random variables \( X_1, \ldots, X_{10} \), each uniform on \([0, 6]\). Let \( Z \) be the number of times \( i \) such that \( X_i < N \) (thus the possible values of \( Z \) are \( 0, \ldots, 10 \)). Compute \( E(Z) \).

2. Assume that a random walk on \( \{0, 1, 2, 3\} \) moves as follows. At 1 and 2, it jumps to the left or to the right with probability 1/2. At 0, it stays there or jumps to the right with probability 1/2. At 3, it stays there or jumps to the left with probability 1/2.

Write down the transition matrix for the resulting Markov chain. Show that the chain is irreducible and aperiodic, and compute the invariant distribution. (Use symmetry to simplify computations!)
3. Peter owns two pairs of running shoes. Each morning he goes running. He is equally likely
to leave from his front or back door. Upon leaving the house, he chooses a pair of running shoes
at the door from which he leaves, or goes running barefoot if there are no shoes there. On his
return, he is equally likely to enter at each door, and leaves his shoes (if any) there.
(a) What proportion of days he runs barefoot?
(b) What proportion of days is there at least one pair of shoes at the front door (before he
going running)?

4. Start at 0 and perform the following random walk on integers. Flip three fair coins and count
the number of heads. If you get no heads, you stay where you are. If you get i heads, you move
i steps to the right. Let $p_n$ be the probability that you ever visit the integer n.
(a) Compute $\lim_{n \to \infty} p_n$.
(b) Compute $p_1$ and $p_2$.

5. In a branching process, the number of descedants is distributed as the following random
variable $X$:

\[ P(X = 0) = \frac{1}{8}, P(X = 1) = \frac{1}{4}, P(X = 2) = \frac{1}{2}, P(X = 3) = \frac{1}{8}. \]

Compute $\pi_0$, the probability that the branching process dies out eventually.

6. Assume that certain events (say power surges) occur as a Poisson process with rate 3 per
hour. These events cause damage to a certain system (say a computer), thus a special protecting
unit has been designed. That unit now has to be removed from the system for 10 minutes for
service.
(a) Assume that a single event occurring in the service period will cause the system to crash.
What is the probability that the system will crash?
(b) Assume that the system will survive a single event, but two events occurring in the service
period will cause it to crash. What is now the probability that the system will crash?