Lecture 3: Permutations

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Permutations

Assume we have a set S of n objects. Choose r elements of S and put them into some order. Let $P(n, r)$ be the number of ways to do this, the number of *r-permutations* of an n-element set.

In particular, $P(n, n)$ is the number of *permutations* of S, the number of ways to order S.

For example, $P(n, 0) = 1$, $P(n, 1) = n$, $P(n, r) = 0$ for $r > n$.

Permutations

Theorem

For $n, r > 0$, $r \leq n$,

$$P(n, r) = n(n-1)\cdots(n-r+1)$$

and so

$$P(n, n) = n(n-1)\cdots 1 = n!$$

and

$$P(n, r) = \frac{n!}{(n-r)!}.$$

Proof.

Choose 1st element in the order (n choices), the the 2nd ($n-1$ choices), ..., and finally the rth ($n-r+1$ choices). $\square$

Note: we define $0! = 1$.

Permutations examples

Example 3.1. Number of different batting orders for 9 baseball players.
Answer: $9! = 362\,880$.

Permutations examples

Example 3.2. A students wants to arrange 5 math books, 3 chemistry books, 2 history books and 2 language books on the shelf.
In how many ways can this be done?
We assume the books are all different, i.e., distinguishable.

Answer: $12! = 479\,001\,600$.

Now the student wants the same-subject books to be together on the shelf. In how many ways can this be done?

Answer: order the 4 subjects, then books within each subject to get $4! \cdot 5! \cdot 3! \cdot 2! \cdot 2! = 69\,120$.

Permutations examples

Example 3.3. A group of 20 Scandinavians includes 6 Swedes and 14 Norwegians. In how many ways can they be seated in a row of 20 chairs so that:
(a) All 6 Swedes sit together (next to each other).
(b) No two Swedes sit together.
(The 20 Scandinavians are distinguishable!)

Permutations examples

A group of 20 Scandinavians includes 6 Swedes and 14 Norwegians. In how many ways can they be seated in a row of 20 chairs so that: (a) All 6 Swedes sit together (next to each other).

Answer to (a): The Swedes must occupy a block of 6 chairs. The block, together with 14 Norwegians makes 15 objects, that are to be ordered, and then the 6 Swedes within the block are also to be ordered. Therefore, the answer is:
$15! \cdot 6! \approx 9.4 \cdot 10^{14}$.

Permutations examples

A group of 20 Scandinavians includes 6 Swedes and 14 Norwegians. In how many ways can they be seated in a row of 20 chairs so that: (b) No two Swedes sit together.

Answer to (b): order Norwegians first (14! ways) and then sit Swedes in-between them ($P(15, 6)$ ways). So the answer is: $14! \cdot 15!/9! \approx 3.14 \cdot 10^{17}$.

Circular permutations

Assume we have n people, and r of them are to be seated on r chairs around a table. Assume that you consider any rotation (a.k.a. cyclic permutation) of a seating arrangement to be the same arrangement. What is the number of such arrangements, the *circular r-permutations*?

Theorem

The number of circular r-permutations of n objects is

$$\frac{1}{r} P(n, r).$$

In particular, the number of circular n-permutations of n objects is $(n-1)!$.

Circular permutations

Proof.

Consider the equivalence relation on r-permutations, whereby two r-permutations are equivalent if they are rotations of each other. The circular r-permutations are exactly the equivalence classes. Each equivalence class has the same number of elements: r. By the division principle, the number of circular r-permutations is $\frac{1}{r}P(n, r)$. $\square$

A more practical way to view the circular n-permutations of n objects is to observe that we can always seat a particular person on a particular chair, by an appropriate rotation. The remaining chairs form a row of $n - 1$ chairs in which to seat $n - 1$ people.

Circular permutations examples

Example 3.3. A group of 20 Scandinavians includes 6 Swedes, 3 Norwegians and 11 Finns. In how many ways can they be seated on 20 chairs around the table so that:
(a) There are no additional restrictions.
(b) All 6 Swedes sit together (next to each other).
(c) All groups sit together.
(d) No two Swedes sit together.

Answer to (a): $19!$.

Circular permutations examples

A group of 20 Scandinavians includes 6 Swedes, 3 Norwegians and 11 Finns. In how many ways can they be seated on 20 chairs around the table so that:
(b) All 6 Swedes sit together (next to each other).

Answer to (b). Pick, say, Magnus, the Norwegian, and sit him in the prescribed chair. Then this becomes a row problem with 6 Swedes and 13 other Scandinavians, and the answer is $14! \cdot 6!$.

“Why me?”

Circular permutations examples

A group of 20 Scandinavians includes 6 Swedes, 3 Norwegians and 11 Finns. In how many ways can they be seated on 20 chairs around the table so that:
(c) All groups sit together.

Answer to (c). Again, Magnus gets seated in the prescribed chair. The Norwegians are ordered in $3!$ ways, which also assigns their seats. We now have a row ordering of the 6 Swedes and 11 Finns, and the two groups need to sit together, for $2! \cdot 6! \cdot 11!$ ways. The answer is

$$3! \cdot 2! \cdot 6! \cdot 11!.$$

Circular permutations examples

Example 3.3. A group of 20 Scandinavians includes 6 Swedes, 3 Norwegians and 11 Finns. In how many ways can they be seated on 20 chairs around the table so that:
(d) No two Swedes sit together.

Answer to (d): Yet again, Magnus gets seated in the prescribed chair. Then we have the row arrangement with 6 Swedes and 13 non-Swedes, and the latter get ordered first (13! ways) followed by placing of Swedes in-between them ($P(14, 6) = 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 = 14!/8!$ ways). The answer is

$$13! \cdot 14!/8!.$$