Lecture 5: Permutations of multisets

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What is a multiset

A multiset is a set of elements, together with positive integer *repetition numbers* assigned to each element. For example,

$$M = \{3 \cdot a, 1 \cdot b, 3 \cdot c, 4 \cdot d\}$$

is the multiset with 3 indistinguishable elements a (3 repetitions of a), 1 b, 3 cs and 4 ds. The repetition numbers could be infinite, e.g.

$$M = \{\infty \cdot a, 1 \cdot b, 2 \cdot c, \infty \cdot d\}$$

The different elements are called *types*.

Permutation of multisets

Assume that n is a total number of elements in a multiset S (counting repetition). Then an r-permutation is again an ordering of r elements from S. Again, an n-permutation is simply called a permutation.

We start with the easiest case.

Theorem

Assume that S has k types, all with infinite repetition numbers. Then the number of r-permutations is k^r.

Example 5.1. Number of 4-digit numbers with all digits odd is 5^4.

Permutation of multisets

Theorem

Assume that S has k types with finite repetition numbers $n_1, \dots, n_k$, so that the number of elements is $n = n_1 + \dots + n_k$. Then the number of permutations of S is

$$\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \binom{n-n_1-n_2}{n_3} \cdots \binom{n-n_1-\dots-n_{k-1}}{n_k}$$
$$= \frac{n!}{n_1! \cdots n_k!},$$

which is sometimes denoted by

$$\binom{n}{n_1\ n_2\ \dots\ n_k}$$

and called a multinomial coefficient.

Permutation of multisets

Proof.

Assume $S = \{n_1 \cdot 1, n_2 \cdot 2, \ldots, n_k \cdot k\}$. Then we obtain a permutation of S by first placing 1s onto n_1 slots chosen from n slots, then placing 2s onto n_2 slots chosen from $n - n_1$ remaining slots, then placing 3s onto n_3 slots chosen from $n - n_1 - n_2$ remaining slots, etc. The number of choices thus is

$$\binom{n}{n_1} \cdot \binom{n - n_1}{n_2} \cdot \binom{n - n_1 - n_2}{n_3} \cdots$$

$$= \frac{n!}{n_1!\cancel{(n - n_1)!}} \cdot \frac{\cancel{(n - n_1)!}}{n_2!\cancel{(n - n_1 - n_2)!}} \cdot \frac{\cancel{(n - n_1 - n_2)!}}{n_3!(n - n_1 - n_2 - n_3)!} \cdots$$

$$= \frac{n!}{n_1! \cdots n_k!}.$$

□

Permutation of multisets: examples

Example 5.2. Assume you have 4 identical cans of red paint, 3 cans of blue paint, 2 cans of yellow paint and one can of green paint. Each can is exactly enough to paint a room. In how many ways can you paint (a) 10 rooms and (b) 9 rooms.

Answer to (a) is given by the theorem:

$$\binom{10}{4} \cdot \binom{6}{3} \cdot \binom{3}{2} \cdot \binom{1}{1} = \frac{10!}{4!3!2!1!} = 12\,600.$$

Permutation of multisets: examples

Answer to (b). *Same* as the last room's color is determined after 9 rooms are painted.

Permutation of multisets: examples

Example 5.3. How many “words” can be made using letters PEPPER?

Answer:

$$\frac{6!}{3!2!1!} = 60.$$

Permutation of multisets: examples

Example 5.4. How many 12-digit numbers have (a) 3 0s, 2 1s, 2 2s, and 5 3s and (b) 2 1s and 3 2s.

Permutation of multisets: examples

Answer to (a). Count first without the requirement that the first digit cannot be zero, then subtract the number of cases with first digit zero:

$$\frac{12!}{3!2!2!5!} - \frac{11!}{2!2!2!5!}.$$

Permutation of multisets: examples

Answer to (b). Proceed as in (a), but now choose positions of 1s and 2s, and then fill in the remaining positions, each with one of remaining 8 digits:

$$\binom{12}{2}\binom{10}{3}8^7 - \binom{11}{2}\binom{9}{3}8^6.$$

Nonattacking rooks

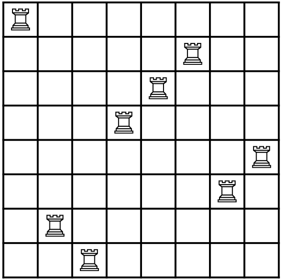
Example 5.5. Standard 8×8 chessboard, 8 rooks. Put them on the board so that they do not attack each other; that is, each row or column contains at most one rook. The extra assumption:

(a) Rooks are indistinguishable.

(b) Rooks are all distinguishable, painted different colors.

(c) Rooks are painted 3 colors: 4 blue, 3 red, 1 green.

In how many ways can you do this?



A placement of 8 nonattacking rooks.

Answer to (a): $8!$.

Nonattacking rooks

Answer to (b). After placing the rooks, paint them: $(8!)^2$.

Answer to (c). Choose rows for each color:

$$\binom{8}{4}\binom{4}{3}8! = \frac{(8!)^2}{4!3!1!}$$