

Lecture 20: Rook polynomials

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Rook polynomial: definition

Assume we have a (finite) set of positions B on a chessboard, labeled by \times 's. Recall that r_k is the number of ways to put k nonattacking rooks on the \times 's. We define $r_0 = 1$, so that r_k is defined for $k = 0, 1, \dots$, but only finitely many of them are nonzero. The *rook polynomial* $r_B(x)$ is the generating function of this sequence:

$$r_B(x) = r_0 + r_1x + r_2x^2 + \dots$$

We want to develop methods to compute rook polynomials that do not depend on tedious counting of all cases, as we did in the following example.

Rook polynomial: example

Example 20.1.

×					
×	×				
		×	×		
		×	×		

Here we have:

$$r_1 = 7$$

$$r_2 = \binom{7}{2} - 6 = 15$$

$$r_3 = 4 + 2 + 2 + 2 = 10$$

$$r_4 = 2$$

So in this example $r_B(x) = 1 + 7x + 15x^2 + 10x^3 + 2x^4$.

Example 20.2. Find the rook polynomial for the full 5×10 board.

In this case,

$$\begin{aligned} r_B(x) = & 1 + 5 \cdot 10x + \binom{5}{2} \binom{10}{2} 2! x^2 + \binom{5}{3} \binom{10}{3} 3! x^3 \\ & + \binom{5}{4} \binom{10}{4} 4! x^4 + \binom{5}{5} \binom{10}{5} 5! x^5 \end{aligned}$$

Rook polynomial: two rows

Note that you can delete empty rows and columns from a board, and also permute rows and columns as convenient.

It is easy to compute rook polynomials for boards with only two rows or columns.

Example 20.3.

	×	×	×		
×	×	×			×

Here we have $r_1 = 7$. To compute r_2 , either subtract the number of all attacking pairs or add, over all placements on the first row, the number of possible placements on the second row:

$$r_2 = \binom{7}{2} - \binom{3}{2} - \binom{4}{2} - 1 - 1 = 10$$

$$r_2 = 3 + 3 + 4$$

$$r_B(x) = 1 + 7x + 10x^2$$

Independent sets of positions

Theorem

Assume that B is partitioned into two parts C and D which have no row or column in common. Then

$$r_B(x) = r_C(x)r_D(x)$$

Example 20.4.

×			
	×		
		×	
			×

$$r_B(x) = (1 + x)^4$$

×	×		
×	×		
		×	×
		×	×

$$r_B(x) = (1 + 4x + 2x^2)^2$$

Independent sets of positions

Proof.

$$\begin{aligned} & \text{coeff. of } x^k \text{ in } r_B(x) \\ &= \sum_{i=0}^k (\text{no. of ways to put } i \text{ rooks on } C \text{ and } k-i \text{ on } D) \\ &= \sum_{i=0}^k (\text{no. of ways to put } i \text{ rooks on } C) \\ & \quad \cdot (\text{no. of ways to put } k-i \text{ rooks on } D) \\ &= \sum_{i=0}^k (\text{coeff. of } x^i \text{ in } r_C(x)) \cdot (\text{coeff. of } x^{k-i} \text{ in } r_D(x)) \\ &= \text{coeff. of } x^k \text{ in } r_C(x) \cdot r_D(x) \end{aligned}$$



Eliminating one position

Theorem

Assume that s is any square of B . Let B_1 be the board obtained by deleting s from B and B_2 the board obtained by deleting the entire row and entire column containing s . Then

$$r_B(x) = r_{B_1}(x) + x r_{B_2}(x).$$

Example 20.5.

$B =$

×	×	×	×
×			
×			
×			

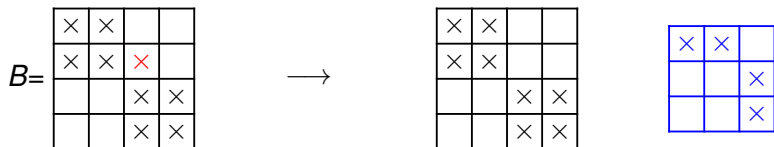
\rightarrow

	×	×	×
×			
×			
×			

\emptyset

$$r_B(x) = (1 + 3x)^2 + x \cdot 1$$

Eliminating one position



$$r_B(x) = (1 + 4x + 2x^2)^2 + x \cdot (1 + 2x)^2$$

Eliminating one position

Proof.

For $k \geq 1$:

$$\begin{aligned} & \text{coeff. of } x^k \text{ in } r_B(x) \\ &= (\text{no. of ways to put } k \text{ rooks on } B \text{ not using } s) \\ & \quad + (\text{no. of ways to put } k \text{ rooks on } B \text{ using } s) \\ &= (\text{no. of ways to put } k \text{ rooks on } B_1) \\ & \quad + (\text{no. of ways to put } k - 1 \text{ rooks on } B_2) \\ &= (\text{coeff. of } x^k \text{ in } r_{B_1}(x)) \\ & \quad + (\text{coeff. of } x^{k-1} \text{ in } r_{B_2}(x)) \\ &= (\text{coeff. of } x^k \text{ in } r_{B_1}(x)) \\ & \quad + (\text{coeff. of } x^k \text{ in } x r_{B_2}(x)) \end{aligned}$$



Rook polynomials: example

Example 20.6. Compute the rook polynomial for the following set B :

		×	×	×	
		×	×	×	
×	×				
	×				×
×					×
	×				×

Then determine the number of ways to place 6 nonattacking rooks on the board off the set B .

First, observe that B can be divided into two independent sets:

×	×	×
×	×	×

×	×	
	×	×
×		×
	×	×

Rook polynomials: example

×	×	×
×	×	×

$$\text{r.p.} = 1 + 6x + 6x^2$$

×	×	
	×	×
×		×
	×	×



	×	
	×	×
×		×
	×	×

×	×
	×
×	×

	×	
	×	×
×		×
	×	×



×	
×	×
	×
×	×

×	
×	×
×	×

×	×
	×
×	×

×	
×	×
×	×

both $1 + 5x + 4x^2$

×	
×	×
	×
×	×

$1 + 6x + 7x^2$

$$r_B(x) =$$

$$(1 + 6x + 6x^2)((1 + 6x + 7x^2) + x(1 + 5x + 4x^2) + x(1 + 5x + 4x^2))$$

Rook polynomials: example

$$\begin{aligned}r_B(x) &= (1 + 6x + 6x^2)((1 + 6x + 7x^2) + 2x(1 + 5x + 4x^2)) \\ &= 1 + 14x + 71x^2 + 158x^3 + 150x^4 + 48x^5\end{aligned}$$

Observe that there is no x^6 . What does this mean?

Rook polynomials: example

It is impossible to place 6 nonattacking rooks *on* the \times 's, even though there is no empty row or column.

		\times	\times	\times	
		\times	\times	\times	
\times	\times				
	\times				\times
\times					\times
	\times				\times

$$r_B(x) = 1 + 14x + 71x^2 + 158x^3 + 150x^4 + 48x^5$$

By our inclusion-exclusion formula, the number of ways to put 6 nonattacking rooks *off* the \times 's then is

$$6! - 14 \cdot 5! + 71 \cdot 4! - 158 \cdot 3! + 150 \cdot 2! - 48 \cdot 1! = 48$$