Lecture 25: Domino tilings

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MAT 145

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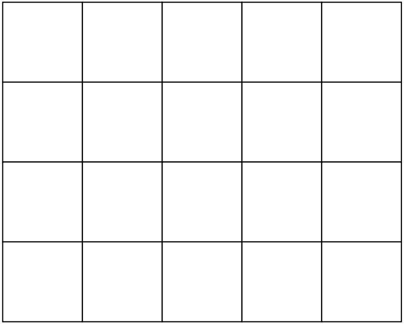
This will not be on the exam!

Domino tilings

Take an $m \times n$ board. Can it be perfectly covered (or *tiled*) by dominoes?



Obviously it can. But let's look a bit closer...

We have a graph: vertices are squares, and edges are between horizontally or vertically adjacent squares.

Domino tilings

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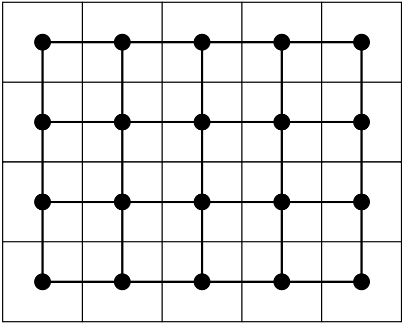
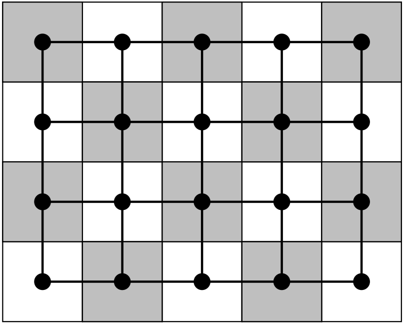


Obviously it can. But let's look a bit closer...

We have a graph: vertices are squares, and edges are between horizontally or vertically adjacent squares.

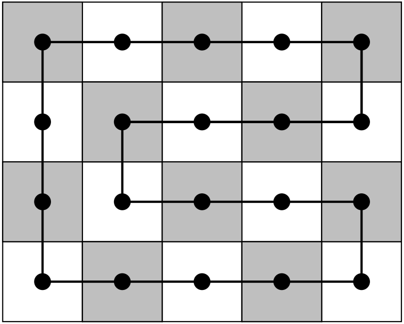
Dominoes and matchings

This is a bipartite graph! If we color the edges as a chessboard, then all edges are between squares of different colors. The bipartite graph has a "black" side and a "white" side.



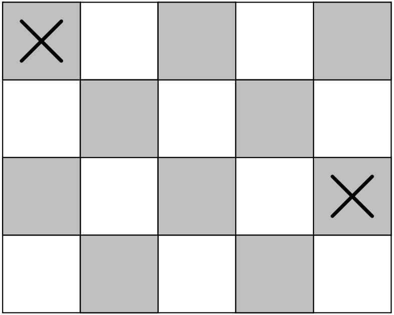
Placing a domino amounts to choosing an edge. Placing some non-overlapping dominoes amounts to choosing some edges without a common vertex, that is, choosing a matching. *Placing non-overlapping dominoes is the same as choosing a matching in this bipartite graph*.

Hamiltonian cycle



The graph also has a Hamiltonian cycle: a sequence of neighboring vertices which visits every vertex once before it returns to the starting vertex.

Now we eliminate two squares as indicated. Can the reduced board still be perfectly covered by dominoes?

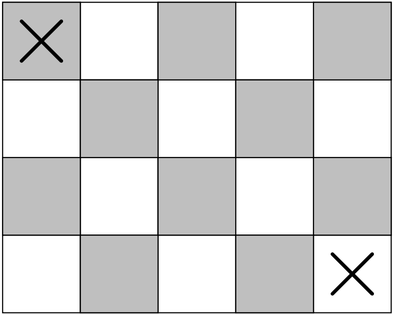


Non-rectangular boards

No! Both eliminated squares are black. The black side of our bipartite graph now has 8 vertices and the white side 10 vertices. We cannot place more that 8 non-overlapping dominoes. (And we indeed can place 8 of them: one in the first column, one in the last column and 6 in the other columns.)

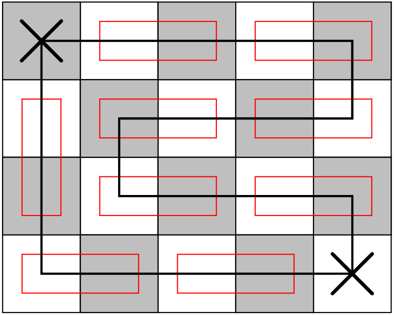
Non-rectangular boards

What if we eliminate a black and a white square?



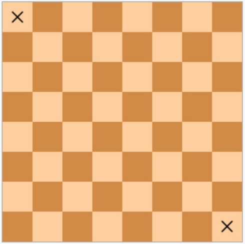
Non-rectangular boards

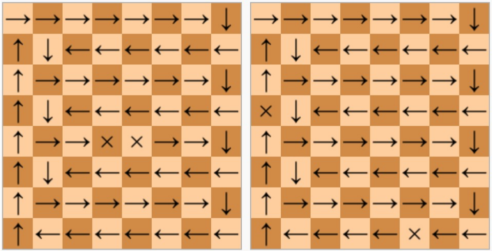
Yes! We use the Hamiltonian path. This works for any two squares of different colors. The two $\times$'s divide the Hamiltonian path into two paths of even length (or just one path of even length, if the two removed squares are neighbors on the Hamiltonian path).



Non-rectangular boards

These problems are known as “mutilated chessboard” puzzles. The top board cannot be perfectly covered by dominoes, but the bottom two can. (Pictures from Wikipedia.)



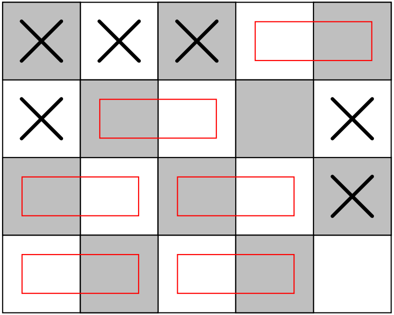


A

B

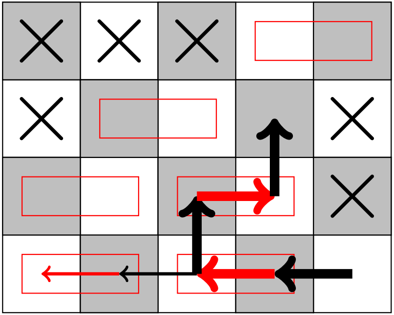
Non-rectangular boards

Do the dominoes below provide a maximal number of non-overlapping dominoes we can place? Equivalently, the question is whether the matching is maximal. We run the matching algorithm.



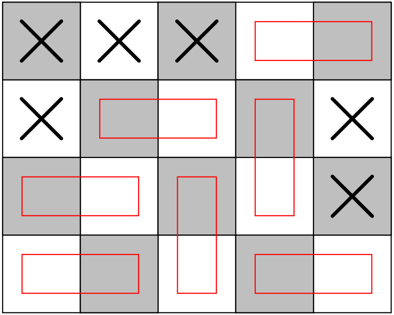
Non-rectangular boards

The algorithm finds an alternating chain. (The arrows indicate which vertex (i.e., square) labels what.)



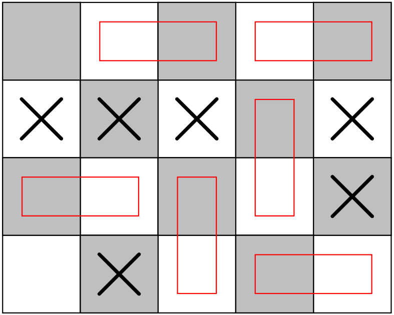
Non-rectangular boards

So the matching is not maximal and there is a perfect cover.



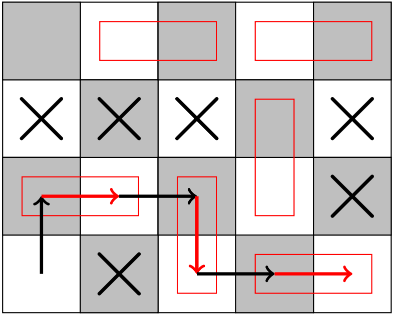
Non-rectangular boards

Same question: do the dominoes below provide a maximal number of non-overlapping dominoes we can place?



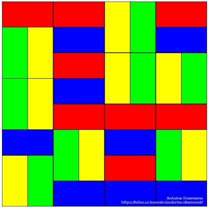
Non-rectangular boards

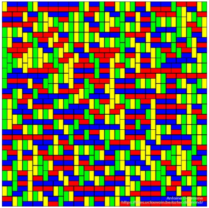
The algorithm ends without finding the free black square. This is a maximal placement of dominoes.



Counting perfect domino tilings

There are exactly 12 988 816 ways to tile the 8×8 chessboard with dominoes. There is a complicated formula for general rectangular boards.

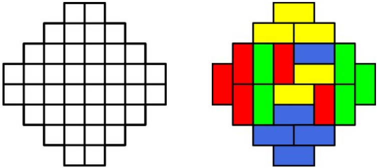




After the tiling is chosen, each domino is painted one of 4 colors, according to whether it is horizontal or vertical and, if horizontal, whether its left square covers a black or a white square on the board (and with the same convention for the top square of a vertical domino).

Aztec diamond

There is a very simple formula for the number of tilings of an Aztec diamond. This is the Aztec diamond with 8 rows and one of its tilings:



If the Aztec diamond has $2n$ rows, then there are exactly

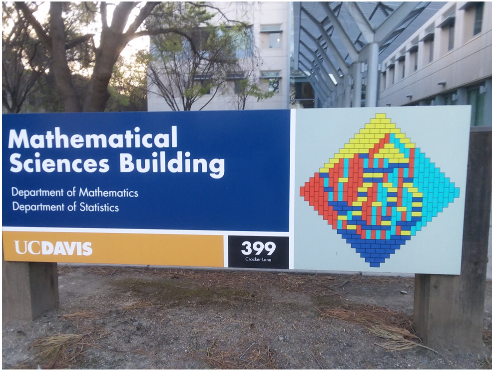
$$2^{n(n+1)/2}$$

tilings.

Aztec diamond

This was proved in 1992 by Noam Elkies, Greg Kuperberg, Michael Larsen, and James Propp. Here is a recent paper with a simpler proof:

Manuel Fendler and Daniel Grieser, *A new simple proof of the Aztec diamond theorem*, paper link



The Arctic circle theorem

Choose a tiling of a large Aztec diamond at random. Then outside of the circle tangent to the four sides, the tiles are regularly arranged, “frozen”. This is a theorem proved in 1998 by William Jockusch, James Propp, and Peter Shor.