

$$\begin{aligned}
&= \sum_{\{e:e|n\}} \left(\sum_{\{r:r|n/e\}} \frac{r}{n} \mu((n/e)/r) \right) k^e \\
&= \sum_{\{e:e|n\}} \frac{\phi(n/e)}{n} k^e \\
&= \frac{1}{n} \sum_{\{e:e|n\}} \phi(n/e) k^e.
\end{aligned}$$

Therefore, the number of circular n -words that can be made from an alphabet of size k equals

$$\frac{1}{n} \sum_{\{e:e|n\}} \phi(n/e) k^e.$$

□

6.7 Exercises

1. Find the number of integers between 1 and 10,000 inclusive that are not divisible by 4, 5, or 6.
2. Find the number of integers between 1 and 10,000 inclusive that are not divisible by 4, 6, 7, or 10.
3. Find the number of integers between 1 and 10,000 that are neither perfect squares nor perfect cubes.
4. Determine the number of 12-combinations of the multiset

$$S = \{4 \cdot a, 3 \cdot b, 4 \cdot c, 5 \cdot d\}.$$

5. Determine the number of 10-combinations of the multiset

$$S = \{\infty \cdot a, 4 \cdot b, 5 \cdot c, 7 \cdot d\}.$$

6. A bakery sells chocolate, cinnamon, and plain doughnuts and at a particular time has 6 chocolate, 6 cinnamon, and 3 plain. If a box contains 12 doughnuts, how many different options are there for a box of doughnuts?
7. Determine the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 14$ in nonnegative integers x_1, x_2, x_3 , and x_4 not exceeding 8.
8. Determine the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 14$ in positive integers x_1, x_2, x_3, x_4 and x_5 not exceeding 5.

9. Determine the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 20$$

that satisfy

$$1 \leq x_1 \leq 6, 0 \leq x_2 \leq 7, 4 \leq x_3 \leq 8, 2 \leq x_4 \leq 6.$$

10. Let S be a multiset with k distinct objects with given repetition numbers n_1, n_2, \dots, n_k , respectively. Let r be a positive integer such that there is at least one r -combination of S . Show that, in applying the inclusion-exclusion principle to determine the number of r -combinations of S , one has $A_1 \cap A_2 \cap \dots \cap A_k = \emptyset$.
11. Determine the number of permutations of $\{1, 2, \dots, 8\}$ in which no even integer is in its natural position.
12. Determine the number of permutations of $\{1, 2, \dots, 8\}$ in which exactly four integers are in their natural positions.
13. Determine the number of permutations of $\{1, 2, \dots, 9\}$ in which at least one odd integer is in its natural position.
14. Determine a general formula for the number of permutations of the set $\{1, 2, \dots, n\}$ in which exactly k integers are in their natural positions.
15. At a party, seven gentlemen check their hats. In how many ways can their hats be returned so that
- no gentleman receives his own hat?
 - at least one of the gentlemen receives his own hat?
 - at least two of the gentlemen receive their own hats?

16. Use combinatorial reasoning to derive the identity

$$\begin{aligned} n! &= \binom{n}{0}D_n + \binom{n}{1}D_{n-1} + \binom{n}{2}D_{n-2} \\ &\quad + \dots + \binom{n}{n-1}D_1 + \binom{n}{n}D_0. \end{aligned}$$

(Here, D_0 is defined to be 1.)

17. Determine the number of permutations of the multiset

$$S = \{3 \cdot a, 4 \cdot b, 2 \cdot c\},$$

where, for each type of letter, the letters of the same type do not appear consecutively. (Thus *abbbbacaca* is not allowed, but *abbacacb* is.)

18. Verify the factorial formula

$$n! = (n-1)((n-2)! + (n-1)!), \quad (n = 2, 3, 4, \dots).$$

19. Using the evaluation of the derangement numbers as given in Theorem 6.3.1, provide a proof of the relation

$$D_n = (n-1)(D_{n-2} + D_{n-1}), \quad (n = 3, 4, 5, \dots).$$

20. Starting from the formula $D_n = nD_{n-1} + (-1)^n$, ($n = 2, 3, 4, \dots$), give a proof of Theorem 6.3.1.

21. Prove that D_n is an even number if and only if n is an odd number.

22. Show that the numbers Q_n of Section 6.5 can be rewritten in the form

$$Q_n = (n-1)! \left(n - \frac{n-1}{1!} + \frac{n-2}{2!} - \frac{n-3}{3!} + \dots + \frac{(-1)^{n-1}}{(n-1)!} \right).$$

23. (Continuation of Exercise 22.) Use the identity

$$(-1)^k \frac{n-k}{k!} = (-1)^k \frac{n}{k!} + (-1)^{k-1} \frac{1}{(k-1)!}$$

to prove that $Q_n = D_n + D_{n-1}$, ($n = 2, 3, \dots$).

24. What is the number of ways to place six nonattacking rooks on the 6-by-6 boards with forbidden positions as shown?

(a)

×	×				
		×	×		
				×	×

(b)

×	×				
×	×				
		×	×		
		×	×		
				×	×
				×	×

(c)

×	×				
	×	×			
		×			
				×	×
					×

25. Count the permutations $i_1 i_2 i_3 i_4 i_5 i_6$ of $\{1, 2, 3, 4, 5, 6\}$, where $i_1 \neq 1, 5$; $i_3 \neq 2, 3, 5$; $i_4 \neq 4$; and $i_6 \neq 5, 6$.
26. Count the permutations $i_1 i_2 i_3 i_4 i_5 i_6$ of $\{1, 2, 3, 4, 5, 6\}$, where $i_1 \neq 1, 2, 3$; $i_2 \neq 1$; $i_3 \neq 1$; $i_5 \neq 5, 6$; and $i_6 \neq 5, 6$.
27. A carousel has eight seats, each representing a different animal. Eight girls are seated on the carousel facing forward (each girl looks at another girl's back). In how many ways can the girls change seats so that each has a different girl in front of her? How does the problem change if all the seats are identical?
28. A carousel has eight seats, each representing a different animal. Eight boys are seated on the carousel but facing inward, so that each boy faces another (each boy looks at another boy's front). In how many ways can the boys change seats so that each faces a different boy? How does the problem change if all the seats are identical?
29. A subway has six stops on its route from its base location. There are 10 people on the subway as it departs its base location. Each person exits the subway at one of its six stops, and at each stop at least one person exits. In how many ways can this happen?
30. How many circular permutations are there of the multiset

$$\{3 \cdot a, 4 \cdot b, 2 \cdot c, 1 \cdot d\},$$

where, for each type of letter, all letters of that type do not appear consecutively?

31. How many circular permutations are there of the multiset

$$\{2 \cdot a, 3 \cdot b, 4 \cdot c, 5 \cdot d\},$$

where, for each type of letter, all letters of that type do not appear consecutively?

32. Let n be a positive integer and let p_1, p_2, \dots, p_k be all the different prime numbers that divide n . Consider the Euler function ϕ defined by

$$\phi(n) = |\{k : 1 \leq k \leq n, \text{GCD}\{k, n\} = 1\}|.$$

Use the inclusion-exclusion principle to show that

$$\phi(n) = n \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right).$$

33. * Let n and k be positive integers with $k \leq n$. Let $a(n, k)$ be the number of ways to place k nonattacking rooks on an n -by- n board in which the positions $(1, 1), (2, 2), \dots, (n, n)$ and $(1, 2), (2, 3), \dots, (n-1, n), (n, 1)$ are forbidden. For example, if $n = 6$ the board is

×	×				
	×	×			
		×	×		
			×	×	
				×	×
×					×

prove that

$$a(n, k) = \frac{2n}{2n-k} \binom{2n-k}{k}.$$

Note that $a(n, k)$ is the number of ways to choose k children from a group of $2n$ children arranged in a circle so that no two consecutive children are chosen.

34. Prove that the convolution product satisfies the associative law: $f * (g * h) = (f * g) * h$.
35. Consider the linearly ordered set $1 < 2 < \dots < n$, and let $F : \{1, 2, \dots, n\} \rightarrow \mathfrak{R}$ be a function. Let the function $G : \{1, 2, \dots, n\} \rightarrow \mathfrak{R}$ be defined by

$$G(m) = \sum_{k=1}^m F(k), \quad (1 \leq k \leq n).$$

Apply Möbius inversion to get F in terms of G .

36. Consider the board with forbidden positions as shown:

	×	×	
×			
			×
	×		

Use formula (6.28) to compute the number of ways to place four nonattacking rooks on this board.

37. Consider the partially ordered set $(\mathcal{P}(X_3), \subseteq)$ of subsets of $\{1, 2, 3\}$ partially ordered by containment. Let a function f in $\mathcal{F}(\mathcal{P}(X))$ be defined by

$$f(A, B) = \begin{cases} 1, & \text{if } A = B, \\ 2, & \text{if } A \subset B \text{ and } |B| - |A| = 1, \\ 1, & \text{if } A \subset B \text{ and } |B| - |A| = 2, \\ -1, & \text{if } A \subset B \text{ and } |B| - |A| = 3. \end{cases}$$

Find the inverse of f with respect to the convolution product.

38. Recall the partially ordered set Π_n of all partitions of $\{1, 2, \dots, n\}$, where the partial order is that of refinement (see Exercise 47 of Chapter 4). Determine the Möbius functions of Π_3 and Π_4 .
39. Let n be a positive integer and consider the partially ordered set $(X_n, |)$, where $X_n = \{1, 2, \dots, n\}$ and the partial order is that of divisibility. Let a and b be positive integers in X_n , where $a|b$. Prove that $\mu(a, b) = \mu(1, b/a)$.
40. Consider the multiset $X = \{n_1 \cdot a_1, n_2 \cdot a_2, \dots, n_k \cdot a_k\}$ of k distinct elements with positive repetition numbers n_1, n_2, \dots, n_k . We introduce a partial order on the combinations of X by stating the following relationship: If $A = \{p_1 \cdot a_1, p_2 \cdot a_2, \dots, p_k \cdot a_k\}$ and $B = \{q_1 \cdot a_1, q_2 \cdot a_2, \dots, q_k \cdot a_k\}$ are combinations of X , then $A \leq B$ provided that $p_i \leq q_i$ for $i = 1, 2, \dots, k$. Prove that this statement defines a partial order on X and then compute its Möbius function.