Math 17C, Spring 2011.
June 6, 2011.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____

ID#: ___________________________________

Instructions: Each of the 6 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a decimal number.

Make sure that you have a total of 10 pages (including this one) with 6 problems.

1	
2	
3	
4	
5	
6	
TOTAL	

$$c_1 e^{\alpha t}\begin{bmatrix} u_1\cos(\beta t) - v_1\sin(\beta t) \\ u_2\cos(\beta t) - v_2\sin(\beta t)\end{bmatrix} + c_2 e^{\alpha t}\begin{bmatrix} u_1\sin(\beta t) + v_1\cos(\beta t) \\ u_2\sin(\beta t) + v_2\cos(\beta t)\end{bmatrix}$$

1. Consider the function f given by $f(x, y) = 3y^2x + 6y^2 + x^3 + 6x^2$.
(a) Compute the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(0, 1, 6)$.

$$\frac{\partial f}{\partial x} = 3y^2 + 3x^2 + 12x = 3(x^2 + y^2 + 4x)$$

$$\frac{\partial f}{\partial y} = 6yx + 12y = 6y(x+2)$$

at $x = 0, y = 1$: $\frac{\partial f}{\partial x} = 3$, $\frac{\partial f}{\partial y} = 12$

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$$z - 6 = 3x + 12(y-1)$$

$$3x + 12y - z - 6 = 0$$

(b) Find all critical points of f.

$\frac{\partial f}{\partial x} = 0$

$\frac{\partial f}{\partial y} = 0$

$y = 0$, $x^2 + 4x = 0$, $x(x+4) = 0$ $(0,0)$, $(-4,0)$

$x = -2$, $4 + y^2 - 8 = 0$, $y^2 = 4$, $(-2,2)$, $(-2,-2)$

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(c) For each critical point, determine whether it is a local maximum, a local minimum, or a saddle point.

$f_{xx} = 6x + 12$

$f_{xy} = 6y$

$f_{yy} = 6x + 12$

$D = f_{xx} f_{yy} - f_{xy}^2$

pt.	D	f_{xx}	
$(0,0)$	$12^2 > 0$	12	local min.
$(-4,0)$	$(-12)^2 > 0$	-12	local max.
$(-2,2)$	$-12^2 < 0$		saddle
$(-2,-2)$	$-(-12)^2 < 0$		saddle

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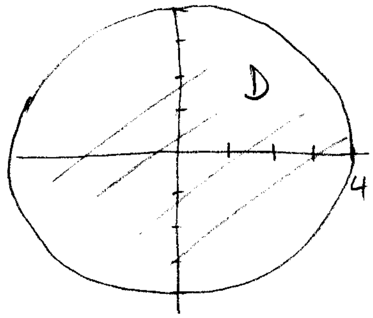
2. Consider the function f given by $f(x, y) = x^2 + y^2 + 4y$.
(a) Determine the global maximum and minimum on $D = \{(x, y) : x^2 + y^2 \leq 16\}$.



$$\frac{\partial f}{\partial x} = 2x \qquad \text{c.p. } \underline{\underline{(0,-2)}}$$

$$\frac{\partial f}{\partial y} = 2y + 4$$

Boundary: $x = 4\cos\theta$, $y = 4\sin\theta$

$g(\theta) = f(4\cos\theta, 4\sin\theta) = 16 + 16\sin\theta$

$g'(\theta) = 16\cos\theta = 0$ at $\theta = \frac{\pi}{3}, \frac{3\pi}{2}$ (only c.p. for $\theta \in [0, 2\pi)$ important here)

1

	pt.	f	
	$(0,-2)$	-4	$\leftarrow$ <u>min</u>
$\theta = \frac{\pi}{2} \rightarrow$	$(0,4)$	32	$\leftarrow$ <u>max</u>
$\theta = \frac{3\pi}{2} \rightarrow$	$(0,-4)$	0	

(b) Determine the direction $\vec{u}$ in which the directional derivative $D_{\vec{u}} f(1, -1)$ is the largest.

This is the direction of $\vec{\nabla} f$ at $(1,-1)$

$$\vec{\nabla} f = \begin{bmatrix} 2x \\ 2y+4 \end{bmatrix}; \text{ at } (1,-1) \quad \vec{\nabla} f = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad |\vec{\nabla} f| = \sqrt{8} = 2\sqrt{2}$$

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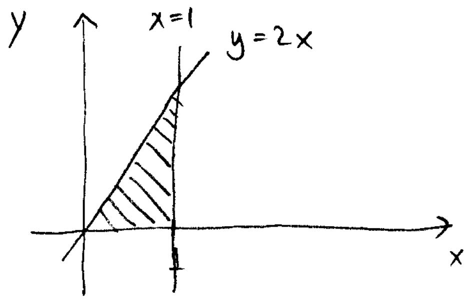
$$\underline{\underline{\vec{u} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}}$$

2, continued ($f(x,y) = x^2 + y^2 + 4y$).
(c) The bounded region D in the plane lies in the first quadrant ($x \geq 0$, $y \geq 0$) and is bounded by the lines $y = 2x$ and $x = 1$. Compute the volume of the spatial region above D and under the surface $z = f(x,y)$. Give the result as a simple fraction.



$$\iint_D f(x,y)\,dx\,dy =$$

$$= \int_0^1 dx \int_0^{2x} (x^2+y^2+4y)\,dy$$

$$= \int_0^1 dx \left[x^2 y + \frac{y^3}{3} + 2y^2\right]_{y=0}^{y=2x}$$

$$= \int_0^1 \left[2x^3 + \frac{8}{3}x^3 + 8x^2\right] dx$$

$$= \int_0^1 \left(\frac{14}{3}x^3 + 8x^2\right) dx$$

$$= \frac{14}{3}\cdot\frac{1}{4} + 8\cdot\frac{1}{3} = \frac{7}{6} + \frac{8}{3} = \underline{\underline{\frac{23}{6}}}$$

3. Consider the linear system

$$\frac{dx_1}{dt} = 2x_1 - 7x_2$$
$$\frac{dx_2}{dt} = 3x_1 - 8x_2$$

(a) Find the general solution of this system.

$$\det \begin{bmatrix} 2-\lambda & -7 \\ 3 & -8-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(-8-\lambda) + 21 = 0$$
$$\lambda^2 + 6\lambda + 5 = 0$$
$$\lambda = -1, -5$$

$\lambda = -1 \qquad 3x_1 - 7x_2 = 0 \qquad \vec{u}_1 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

$\lambda = -5 \qquad 7x_1 - 7x_2 = 0 \qquad \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

~~8~~ 7

Gen. sol:

$$\vec{x} = c_1 e^{-t} \begin{bmatrix} 7 \\ 3 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) Assume that the solution is at the point $(2, 1)$ at time $t = 0$ and *carefully* draw its trajectory, backward and forward in time. Also indicate the direction of this trajectory at time $t = 0$.

$$7c_1 + c_2 = 2$$
$$3c_1 + c_2 = 1$$

$4c_1 = 1, \quad c_1 = \frac{1}{4}$

$c_2 = 1 - \frac{3}{4} = \frac{1}{4}$

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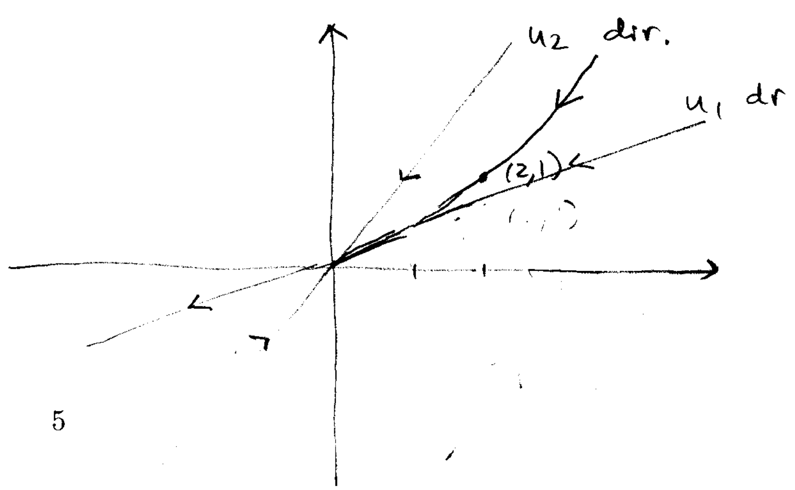
$$\vec{x} = \frac{1}{4} e^{-t} \begin{bmatrix} 7 \\ 3 \end{bmatrix} + \frac{1}{4} e^{-5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

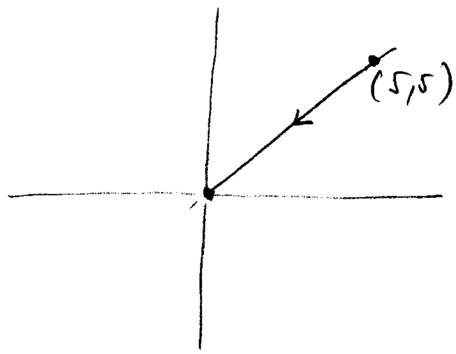
At $t=0$:
$(x_1 = 2, x_2 = 1)$ $\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$



3, continued.

(c) Now assume that the solution is at the point $(5, 5)$ at time $t = 0$. Describe and draw the trajectory.

The trajectory moves in a straight line towards 0.



$$\vec{x} = 5e^{-5t}\begin{bmatrix}1\\1\end{bmatrix}$$

(d) Assume that now $\dfrac{dx_1}{dt} = 2x_1 - 7x_2 + x_1^5$ while $\dfrac{dx_2}{dt} = 3x_1 - 8x_2$ is the same as before. Classify the fixed point $(0, 0)$ and discuss its stability.

The Jacobian at $x_1 = x_2 = 0$:

$$\frac{\partial f_1}{\partial x_1} = 2 + 5x_1^4 \qquad \frac{\partial f}{\partial x_2} = -7$$

$$\frac{\partial f_2}{\partial x_1} = 3 \qquad \frac{\partial f_2}{\partial x_2} = -8$$

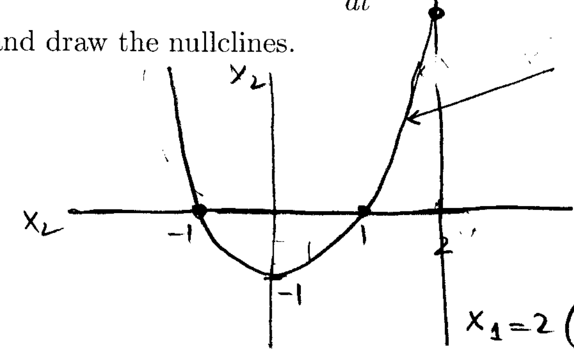
6

$$J = \begin{bmatrix} 2 & -7 \\ 3 & -8 \end{bmatrix}$$ is the matrix of the linear system in (a), so $(0,0)$ is a <u>stable</u> <u>node</u> (or <u>sink</u>).

4. Consider the nonlinear system

$$\frac{dx_1}{dt} = x_1^2 - x_2 - 1$$
$$\frac{dx_2}{dt} = (x_1 - 2)x_2 \quad = x_1 x_2 - 2x_2$$

(a) Determine and draw the nullclines.



(b) Assume that a solution starts (i.e., at $t = 0$) at the point $(1, 2)$. Determine its initial speed and direction of motion.

$$x_1 = 1,\ x_2 = 2: \quad \frac{dx_1}{dt} = -2, \quad \frac{dx_2}{dt} = -2$$

Velocity vector $\begin{bmatrix}-2\\-2\end{bmatrix}$; speed $= \sqrt{4+4} = \sqrt{8} = \underline{\underline{2\sqrt{2}}}$

Direction $= \frac{1}{\sqrt{8}}\begin{bmatrix}-2\\-2\end{bmatrix} = \underline{\underline{\begin{bmatrix}-1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}}}$

(c) Find, classify, and discuss the stability of all of its fixed points.

$x_2 = 0, \quad x_1^2 - 1 = 0, \quad x_1 = \pm 1$ $\qquad (1, 0)$, $(-1, 0)$

$x_1 = 2, \quad x_2 = x_1^2 - 1 = 3$ $\qquad (2, 3)$

$$J = \begin{bmatrix} 2x_1 & -1 \\ x_2 & x_1 - 2 \end{bmatrix}$$

At $(1,0)$: $\begin{bmatrix} 4 & -1 \\ 0 & -1 \end{bmatrix}$ eigenvalues $4, -1$ — saddle (unstable)

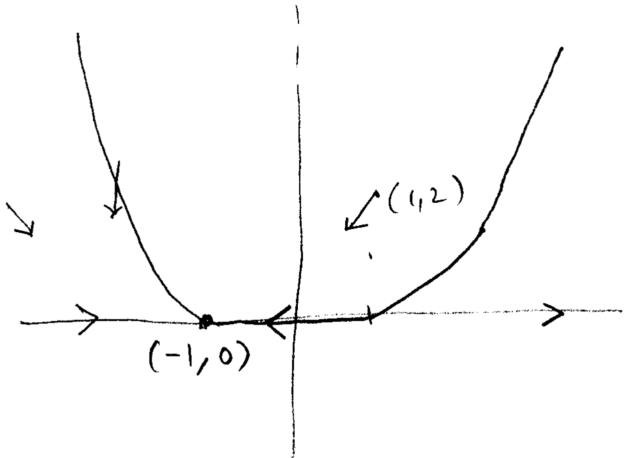
At $(-1,0)$: $\begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix}$ eigenvalues $-2, -3$ — stable node (sink)

At $(2,3)$: $\begin{bmatrix} 4 & -1 \\ 3 & 0 \end{bmatrix}$ $\quad 0 = (4-\lambda)(-\lambda) + 3 = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$

eigenvalues $3, 1$ — unstable node (source)

4, continued.

(d) What happens with the trajectory of the solution in (b) as t goes to infinity?



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The trajectory has to move down and towards (-1,0), so it converges to (-1, 0).

6. A test for certain disease is far from perfect: if you have a disease it gives a positive result (i.e., indicates you have the disease) with probability 0.7, and if you do not have a disease it gives a positive result with probability 0.1. Assume different tests are independent. It is known that 5% of the population has the disease.

(a) Assume that you do not have the disease, and are tested four times. What is the probability that none of the tests are positive?

$$0.9^4$$

6

0.9: 3

(b) Under the same assumption as in (a), what is the probability that exactly one test is positive?

$$4 \cdot 0.9^3 \cdot 0.1$$

(no 4: 3)

6

0.95: 3

+ 1

(c) A random person is chosen from the population, tested twice, and both tests is positive. What is the probability that he/she has the disease?

$A = \{2+ \text{ tests}\}$

$B_1 = \{\text{has disease}\}$ $\quad P(B_1) = 0.05 \quad P(A|B_1) = 0.7^2$

$B_2 = \{\text{does not have dis.}\}$ $\quad P(B_2) = 0.95 \quad P(A|B_2) = 0.1^2$

$$P(B_1|A) = \frac{0.05 \cdot 0.7^2}{0.05 \cdot 0.7^2 + 0.95 \cdot 0.1^2}$$

7

(no sq: 4)

(add term in both: 3

(d) A random person is chosen from the population, tested twice, and both tests are positive. He is going to be tested for the third time. What is the probability that the third test will be positive?

$$P(3+ \text{ tests} \mid 2+ \text{ tests}) = \frac{0.05 \cdot 0.7^3 + 0.95 \cdot 0.1^3}{0.05 \cdot 0.7^2 + 0.95 \cdot 0.1^2}$$

4

<u>or</u> (same ans.)

$$(\text{prob. in (c)}) \cdot 0.7 + (1 - \text{prob. in (c)}) \cdot 0.1$$

not putting together: 3

top: 2

5. Initially, a bag contains 11 golf balls: 6 red and 5 blue. Here are the two things you do at each step: you select one of the balls from the bag at random; if your selected ball is red, you return it to the bag, but if it is ~~[illegible]~~ blue, you do not return it to the bag.

(a) Perform 4 described steps. Compute the probability that the first ball selected is blue, the second is red, the third is blue and the fourth is again blue.

$$\frac{5}{11} \cdot \frac{6}{10} \cdot \frac{4}{10} \cdot \frac{3}{9}$$

6

(b) Perform 2 described steps. Compute the probability that the second selected ball is blue.

$$\underbrace{\frac{5}{11} \cdot \frac{4}{10}}_{\text{both blue}} + \underbrace{\frac{6}{11} \cdot \frac{5}{11}}_{\text{red, then blue}}$$

5

(c) Perform 2 described steps. You have made a second selection and it is a blue ball but you have no memory of your first selection. What is the probability that your first selection was also blue?

$$\frac{\frac{5}{11} \cdot \frac{4}{10}}{\frac{\cancel{5}}{\cancel{11}} \cdot \frac{4}{10} + \frac{6}{11} \cdot \frac{5}{11}} = \frac{4 \cdot 11}{4 \cdot 11 + 6 \cdot 10} = \frac{11}{11+15} = \frac{11}{26}$$

6

(d) Now you perform a single step. After that, your friend selects 4 balls at random (without replacement) from the bag. What is the probability that your friend selects 2 red and 2 blue balls?

$B_1 = \{\text{you select red}\}$ $\quad P(B_1) = \frac{6}{11}$ $\quad P(A|B_1) = \frac{\binom{6}{2}\binom{5}{2}}{\binom{11}{4}}$

$B_2 = \{\text{you select blue}\}$ $\quad P(B_2) = \frac{5}{11}$ $\quad P(A|B_2) = \frac{\binom{6}{2}\binom{4}{2}}{\binom{10}{4}}$ 7

Answer: $P(A) = \frac{6}{11} \frac{\binom{6}{2}\binom{5}{2}}{\binom{11}{4}} + \frac{5}{11} \cdot \frac{\binom{6}{2}\binom{4}{2}}{\binom{10}{4}}$

was $\binom{4}{2}$: 3

no $\binom{11}{4}$: 2

indep: 1