Math 17C, Spring 2011.
May 11, 2011.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________________

ID#: ______________________

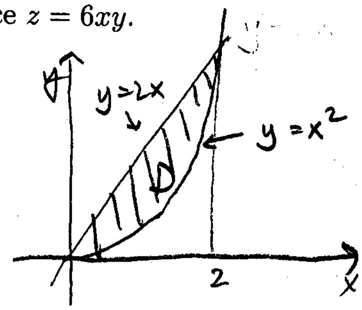
Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

$$c_1 e^{\alpha t} \begin{bmatrix} u_1 \cos(\beta t) - v_1 \sin(\beta t) \\ u_2 \cos(\beta t) - v_2 \sin(\beta t) \end{bmatrix} + c_2 e^{\alpha t} \begin{bmatrix} u_1 \sin(\beta t) + v_1 \cos(\beta t) \\ u_2 \sin(\beta t) + v_2 \cos(\beta t) \end{bmatrix}$$

1. The bounded region D in the plane lies in the first quadrant ($x \geq 0$, $y \geq 0$) and is bounded by the graphs of $y = 2x$ and $y = x^2$. Compute the volume of the spatial region above D and under the surface $z = 6xy$.



$$\text{Volume} = \int_0^2 dx \int_{x^2}^{2x} 6xy\, dy$$

$$= \int_0^2 dx \; 3x \int_{x^2}^{2x} 2y\, dy$$

$$= \int_0^2 dx \cdot 3x \cdot \left. y^2 \right|_{y=x^2}^{y=2x}$$

$$= \int_0^2 3x(4x^2 - x^4)\, dx$$

$$= \int_0^2 (12x^3 - 3x^5)\, dx$$

$$= \left. 3x^4 - \frac{x^6}{2} \right|_0^2 = 3 \cdot 2^4 - 2^5$$

$$= 2^4 = \underline{\underline{16}}$$

2. The position (x_1, x_2) of a particle at time t is given by $x_1 = t^2 + 1$, $x_2 = t^3 - 9t^2 + 24t - 96$. Assume that time is nonegative, i.e., $t \geq 0$.

(a) Determine the speed and the unit vector in the direction of motion at time $t = 1$.

$$\frac{dx_1}{dt} = 2t$$

$$\frac{dx_2}{dt} = 3t^2 - 18t + 24$$

At $t=1$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

$$\text{Speed} = \sqrt{4+81} = \sqrt{85}$$

$$\text{direction} = \frac{1}{\sqrt{85}}\begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{85} \\ 9/\sqrt{85} \end{bmatrix}.$$

9

(b) At which times $t \geq 0$ is the direction of motion horizontal (i.e., in the x_1-direction)? What are the locations of the particle at these times (note: $96 = 4 \cdot 24 = 2 \cdot 48$)?

$$\frac{dx_2}{dt} = 0 \qquad 3t^2 - 18t + 24 = 0$$

$$t^2 - 6t + 8 = 0 \qquad (t-2)(t-4) = 0$$

$\boxed{t=2, \ (5,-76)}$ $\quad x_1 = 5, \ x_2 = 8 - 36 + 48 - 96 = -76$

$\boxed{t=4, \ (17,-80)}$ $\quad x_1 = 17, \ x_2 = 64 - 9 \cdot 16 = -80$

8

(c) Does the particle ever return to a previous position?

No: $\frac{dx_1}{dt} > 0$ so the particle always moves to the right (x_1 always increases).

8

3. Consider the linear system

$$\frac{dx_1}{dt} = x_1 - 4x_2$$
$$\frac{dx_2}{dt} = x_1 + x_2$$

(a) Find the general solution of this system.

$$A = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} \qquad \det(A - \lambda I) = (1-\lambda)^2 + 4$$
$$= \lambda^2 - 2\lambda + 5$$
$$\lambda = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

$\lambda_1 = 1 + 2i$

$$\begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0 \qquad z_1 - 2iz_2 = 0, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2i \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{u}} + i\underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\vec{v}}$$

General solution

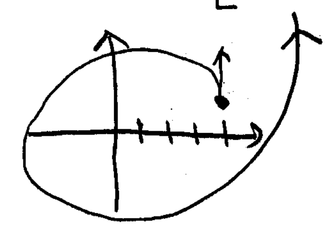
$$\vec{x} = c_1 e^t \begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 e^t \begin{bmatrix} 2\cos(2t) \\ \sin 2t \end{bmatrix}$$ 10

(b) Find the solution that is at the point $(4, 1)$ at time $t = 0$ and sketch is trajectory.

$$c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$2c_2 = 4$, $\underline{c_2 = 2}$, $\vec{x} = e^t \begin{bmatrix} -2\sin(2t) \\ \cos 2t \end{bmatrix} + 2e^t \begin{bmatrix} 2\cos(2t) \\ \sin(2t) \end{bmatrix}$

$\underline{c_1 = 1}$



10

(c) Find, classify, and discuss the stability of the only fixed fixed point of this system.

The only fixed pt. $(0,0)$ is an <u>unstable spiral</u>. 5

4. Consider the nonlinear system

$$\frac{dx_1}{dt} = -x_1 + 2x_2$$
$$\frac{dx_2}{dt} = 2x_1 - x_2^2 + 5$$

Find, classify, and discuss the stability of all of its fixed points. (You may use $\sqrt{11^2 - 24} \approx 9.85$.)

$-x_1 + 2x_2 = 0 \qquad x_1 = 2x_2$

$4x_2 - x_2^2 + 5 = 0, \qquad x_2^2 - 4x_2 - 5 = 0$

$(x_2 + 1)(x_2 - 5) = 0$

Fixed pt.: $(-2, -1)$, $(10, 5)$. 5

$$J = \begin{bmatrix} -1 & 2 \\ 2 & -2x_2 \end{bmatrix}$$

$\underline{(-2, -1)}$

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$$

$(-1-\lambda)(2-\lambda) - 4 = 0$ 10

$\lambda^2 - \lambda - 6 = 0$

$(\lambda - 3)(\lambda + 2) = 0 \qquad \lambda = 3, -2$

<u>saddle</u> (<u>unstable</u>)

$(10, 5)$

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -10 \end{bmatrix}$$

$(-1-\lambda)(-10-\lambda) - 4 = 0$

$\lambda^2 + 11\lambda + 6 = 0$

$$\lambda = \frac{-11 \pm \sqrt{11^2 - 24}}{2}$$ 10

$$\approx \frac{-11 \pm 9.85}{2}$$ both < 0

<u>stable node</u>