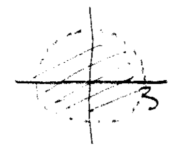
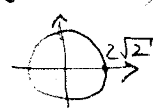
Practice exam, brief solutions

① $9-x^2-y^2>0$, $x^2+y^2<9$

Interior, but not boundary, of the circle with radius 3 and center $(0,0)$



$f(x,y)=0$ $\quad 9-x^2-y^2=1$, $x^2+y^2=8$. Circle of radius $\sqrt{8}$ and center $(0,0)$



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② $\dfrac{\partial z}{\partial x}=2xy+y^2$, $\dfrac{\partial z}{\partial y}=x^2+2xy$

$\dfrac{dx}{dt}=3t^2$, $\dfrac{dy}{dt}=\dfrac{1}{2\sqrt{t}}$

When $t=1$: $x=1$, $y=1$, $\dfrac{\partial z}{\partial x}=3$, $\dfrac{\partial z}{\partial y}=3$, $\dfrac{dx}{dt}=3$, $\dfrac{dy}{dt}=\dfrac{1}{2}$.

Answer: $\dfrac{\partial f}{\partial x}\cdot\dfrac{dx}{dt}+\dfrac{\partial f}{\partial y}\cdot\dfrac{dy}{dt}=3\cdot 3+3\cdot\dfrac{1}{2}=\underline{\underline{\dfrac{21}{2}}}$.

③ $\vec{u}=\begin{bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$, $\nabla f=\begin{bmatrix}\dfrac{2}{2x+3y}\\ \dfrac{3}{2x+3y}\end{bmatrix}$; at $(2,0)$, $\nabla f=\begin{bmatrix}1/2\\ 3/4\end{bmatrix}$,

$D_{\vec{u}}f=-\dfrac{1}{\sqrt{2}}\cdot\dfrac{1}{2}+\dfrac{1}{\sqrt{2}}\cdot\dfrac{3}{4}=\dfrac{1}{4\sqrt{2}}$

④ $\nabla f=\begin{bmatrix}y^3\\ 3xy^2\end{bmatrix}$; at $(2,1)$, $\nabla f=\begin{bmatrix}1\\ 6\end{bmatrix}$ (with length $\sqrt{1+36}=\sqrt{37}$).

(a) $\begin{bmatrix}1/\sqrt{37}\\ 6/\sqrt{37}\end{bmatrix}$ (b) $\begin{bmatrix}-1/\sqrt{37}\\ -6/\sqrt{37}\end{bmatrix}$ (c) $\nabla f\cdot\vec{u}=0$, $u_1+6u_2=0$,

normalize $\begin{bmatrix}-6\\ 1\end{bmatrix}$ to get $\begin{bmatrix}-6/\sqrt{37}\\ 1/\sqrt{37}\end{bmatrix}$

(or its negative).

(d) $\nabla f\cdot\vec{u}=1$

$u_1+6u_2=1$, $u_1^2+u_2^2=1$, $(1-6u_2)^2+u_2^2=1$, $37u_2^2-12u_2=0$,

$u_2=0$, $u_1=1$

$u_2=\dfrac{12}{37}$, $u_1=\dfrac{-35}{37}$

$\begin{bmatrix}1\\ 0\end{bmatrix}$ or $\begin{bmatrix}12/37\\ -35/37\end{bmatrix}$

(5) $\frac{\partial f}{\partial x} = y^2 + 2$

$\frac{\partial f}{\partial y} = 2xy - 3$

At $(0,0)$ $\quad \frac{\partial f}{\partial x} = 2$

$\frac{\partial f}{\partial y} = -3$

$L(x,y) = 2x - 3y$; plug in $x = 0.2$, $y = -0.1$ to get $\underline{0.7}$.

(6) $\frac{\partial z}{\partial x} = 6xy - 12x = 6x(y-2) = 0 \qquad x = 0$ or $y = 2$

$\frac{\partial z}{\partial x} = 3x^2 + 3y^2 - 12y = 3(x^2 + y^2 - 4y) = 0$

$x = 0$ gives $y^2 - 4y = 0$, $y(y-4) = 0$, $y = 0, y = 4$

$\underline{(0,0)}$ and $\underline{(0,4)}$

$y = 2$ gives $x^2 + 4 - 8 = 0$, $x^2 = 4$, $x = \pm 2$

$\underline{(-2,2)}$ and $\underline{(2,2)}$

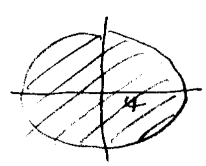
$f_{xx} = 6y - 12$

$f_{yy} = 6y - 12$

$f_{xy} = 6x$

	D	f_{xx}	
(0,0)	144	−12	local max
(0,4)	144	12	local min
(−2,2)	−144		saddle
(2,2)	−144		saddle

(7) $f(x,y) = x^2 + y^2 - 6y + 3$



$\frac{\partial f}{\partial x} = 2x$

$\frac{\partial f}{\partial y} = 2y - 6$

The only interior c.p. is $(0,3)$, $\underline{f(0,3) = -6}$. ← $\underline{\text{MIN}}$

Boundary : $x = 4\cos\theta$, $y = 4\sin\theta$, $0 \le \theta \le 2\pi$

$g(\theta) = f(4\cos\theta, 4\sin\theta) = 19 - 24\sin\theta$

$g'(\theta) = -24\cos\theta \qquad g'(\theta) = 0$ at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\theta = \frac{\pi}{2}$, $(x,y) = (0,4)$, $f(0,4) = -5$

$\theta = \frac{3\pi}{2}$, $\underline{(x,y) = (0,-4), f(0,-4) = 43}$ ← $\underline{\text{MAX}}$