Math 189, Winter 2008.

Problem Set 11: Probability

Below are a few basic strategies for solving probability problems.

- *Computation*: usually the probability space consists of equally likely outcomes; out of these, isolate and count those which correspond to the "good" events. Remember that in the continuous setting, counting corresponds to integrating.
- *Conditioning*: in a sequence of random events, probabilities on one stage often depend on what happens on the previous stage.
- *Recursion*: have this in mind if the problem describes a long sequence of random events (especially if conditioning is useful).
- *Symmetry*: sometimes, probabilities must be equal for this reason.
- *Summation tricks*: for computing expectation of a nonnegative integer valued random variable N, you may use $EN = \sum_{n=1}^{\infty} P(N \geq n)$; for an arbitrary random variable N written in the form $N = \sum_i N_i$ one has $EN = \sum_i EN_i$, if either the sum is finite or $N_i \geq 0$ for all i.

When the distribution of your random choice is not specified, assume it is uniform (that is, all choices are equally likely). If more that one random choice is made, your default assumption should be that the different choices are independent.

1. A dealer takes, at random, two cards out a standard deck of 52 cards. What is the probability that both of your cards are red if:

(1) You are given no additional information;

(2) The dealer looks at *one* of your cards and tells you that it is red;

(3) The dealer looks at *both* of your cards and tells you that at least one is red;

(4) The dealer looks at *both* of your cards and tells you that at least one is a heart;

(5) The dealer looks at *both* of your cards and tells you that one of your cards is the ace of hearts.

2. An unfair coin has probability p of heads. Toss it n times. What is the probability that the number of heads is even?

3.(*) Two real numbers X and Y are chosen at random in the interval $(0, 1)$. Compute the probability that the closest integer to X/Y is even. Express the answer in the form $r + s\pi$, where r and s are rational numbers.

4.(*) Start with n strings, which of course have $2n$ ends. Then randomly pair the ends and tie together each pair. (Therefore you join each of the n randomly chosen pairs.) Let L be the number of resulting loops. Compute $E(L)$.

5. Assume C and D are chosen at random from $\{1, \ldots, n\}$. Let p_n be the probability that $C+D$ is a perfect square. Compute $\lim_{n\to\infty}(\sqrt{n} \cdot p_n)$. Express the result in the form $(a\sqrt{b}+c)/d$, where a, b, c, d are integers.

6. Let $\alpha \in [0,1]$ be an arbitrary number, rational or irrational. The only randomizing device is an unfair coin, with probability $p \in (0,1)$ of heads. Design a game between Alice and Bob so that Alice's winning probability is exactly α. The game of course has to end in a finite number of tosses with probability 1. (The original Putnam formulation had $\alpha = 0.5$.)

7. (*) Let $(X_1, \ldots, X_n)$ be a random vector from the set $\{(x_1, \ldots, x_n) : 0 < x_1 < \cdots < x_n < 1\}$. Also let f be a continuous function on $[0,1]$. Set $X_0 = 0$. Let R be the Riemann sum

$$R = \sum_{i=0}^{n-1} f(X_{i+1})(X_{i+1} - X_i).$$

Show that $ER = \int_0^1 f(t)P(t)\,dt$, where $P(t)$ is a polynomial of degree n, independent of f, with $0 \le P(t) \le 1$ for $t \in [0,1]$.

8. You have $n > 1$ numbers $0, 1, \ldots, n-1$ arranged on a circle. A random walker starts at 0 and at each step moves at random to one of its two nearest neighbors. For each i, compute the probability p_i that, when the walker is at i for the first time, all other points have been previously visited, i.e., that i is the last new point. For example, $p_0 = 0$.

9. Choose $X_1, \ldots, X_n$ from $[0,1]$. Let p_n be the probability that $X_i + X_{i+1} \le 1$ for all $i = 1, \ldots, n-1$. Compute $\limsup_{n\to\infty} p_n^{1/n}$.

10. Each of the triples (r_i, s_i, t_i), $i = 1, \ldots, n$, is a randomly chosen permutation of (1, 2, 3) and each triple is chosen independently. Compute the three sums $\sum_{i=1}^n r_i$, $\sum_{i=1}^n s_i$, and $\sum_{i=1}^n t_i$, and label them (not necessarily in order) A, B, C so that $A \le B \le C$. Let a_n be the probability that $A = B = C$ and let b_n be the probability that $B = A+1$ and $C = B+1$. Show that for every $n \ge 1$, either $4a_n \le b_n$ or $4a_{n+1} \le b_{n+1}$.

Note. This is a reformulation of a Putnam problem, which featured a nasty misdirection. Namely, the last sentence asked for a proof that $4a_n \le b_n$ for some $n \ge 1995$, but the property has nothing to do with large n.

11.(*) Four points are chosen on the unit sphere. What is the probability that the origin lies inside the tetrahedron determined by the four points?

12. An $m \times n$ checkerboard is colored randomly: each square is randomly painted white or black. We say that two squares, p and q, are in the same *connected monochromatic component* (or *component*, in short) if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of components is greater than $mn/8$ and smaller than $(m+2)(n+2)/6$.

13. Choose, at random, three points on the circle $x^2 + y^2 = 1$. Interpret them as cuts that divide the circle into three arcs. Compute the expected length of the arc that contains the point $(1, 0)$.

Remark. Here is a "solution." Let L_1, L_2, L_3 be the lengths of the three arcs. Then $L_1 + L_2 + L_2 = 2\pi$ and by symmetry $E(L_1) = E(L_2) = E(L_3)$, so the answer is $E(L_1) = 2\pi/3$. Explain why this is wrong.