Math 180, Winter 2007.

## Problem Set 6: Inequalities

I will start with a list of famous "elementary" inequalities. The emphasis is on *algebraic* methods rather than calculus. Although I've included a few words about proofs, you really should read the book "The Cauchy-Schwarz Master Class," by J. Michael Steele (Cambridge, 2004) for deeper understanding.

**Jensen.** *Let* $I \subset \mathbb{R}$ *be an interval, and* $f : I \to \mathbb{R}$ *convex. If* $\lambda_1, \ldots, \lambda_n \geq 0$, $\lambda_1 + \cdots + \lambda_n = 1$, *and* $x_1, \ldots, x_n \in I$, *then* $\lambda_1 f(x_1) + \cdots + \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \cdots + \lambda_n x_n)$.

It follows that the maximum of a convex function on $I$ must occur on the boundary of $I$.

**Weighted Power Mean.** *If* $\lambda_1, \ldots, \lambda_n \geq 0$, $\lambda_1 + \cdots + \lambda_n = 1$, *and* $x_1, \ldots, x_n \geq 0$, *then*

$$M(r) = \begin{cases} (\lambda_1 x_1^r + \cdots + \lambda_n x_n^r)^{1/r}, & \text{if } r \neq 0, \\ x_1^{\lambda_1} \cdot \cdots \cdot x_n^{\lambda_n}, & \text{if } r = 0. \end{cases}$$

*is an increasing continuous function of* $r \in \mathbb{R}$. *Moreover,* $M$ *is strictly increasing unless all* $x_i$ *are equal.*

For $r = -1, 0, 1, 2$ and all $\lambda_i = 1/n$, $M(r)$ is, respectively, the harmonic mean (HM), geometric mean (GM), arithmetic mean (AM) and quadratic mean (QM) of $x_i$.

This follows from Jensen for $f(x) = x^\alpha$, and for appropriately chosen $\alpha > 1$, and taking appropriate limits.

**Hölder.** *If* $p, q > 0$, $1/p + 1/q = 1$, *and* $x_i, y_i > 0$, *then*

$$x_1^{1/p} y_1^{1/q} + \cdots + x_n^{1/p} y_n^{1/q} \leq (x_1 + \cdots + x_n)^{1/p} (y_1 + \cdots + y_n)^{1/q}$$

*Equality holds only when* $y_i/x_i$ *is a constant.*

When $p = q = 2$, this is the *Cauchy-Schwarz* inequality. The proof is an application of weighted AM-GM.

**Rearrangement.** *Let* $a_1 \leq \cdots \leq a_n$ *and* $b_1 \leq \cdots \leq b_n$ *(not necessarily positive). Then for any permutation* $\pi$ *of* $\{1, \ldots, n\}$, *we have*

$$a_1 b_1 + \cdots + a_n b_n \geq a_1 b_{\pi(1)} + \cdots + a_n b_{\pi(n)} \geq a_1 b_n + \cdots + a_n b_1.$$

*When* $b_i$ *are strictly increasing, the equality on the left (resp. right) holds only if* $\pi$ *is the identity (resp. reversal).*

The proof is a simple check for transpositions, and then follows for arbitrary $\pi$.

**Chebyshev.** *Under the same assumptions as above,*

$$\frac{a_1 + \cdots + a_n}{n} \cdot \frac{b_1 + \cdots + b_n}{n} \leq \frac{a_1 b_1 + \cdots + a_n b_n}{n}$$

The proof follows from application of Rearrangment for all $n$ cyclic permutations and adding the resulting inequalities.

Next we present a more sophisticated inequality, which can handle a lot of *homogeneous symmetric* inequalities quite easily.

Let $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^n$, with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$. We say that $\alpha$ majorizes $\beta$ if $\alpha_1 + \cdots + \alpha_r \geq \beta_1 + \cdots + \beta_r$ for $1 \leq r \leq n$, with equality when $r = n$. For $x = (x_1, \ldots, x_n)$, with positive coordinates, we define

$$x^\alpha = \frac{1}{n!} \sum_\pi x_{\pi(1)}^{\alpha_1} \cdots x_{\pi(n)}^{\alpha_n},$$

where the sum is over all permutations $\pi$. For example, when $n = 3$,

$$\begin{aligned} x^{(1,0,0)} &= \frac{1}{3}(x_1 + x_2 + x_3), \\ x^{(\frac{1}{2},\frac{1}{2},0)} &= \frac{1}{3}(\sqrt{x_1x_2} + \sqrt{x_1x_3} + \sqrt{x_2x_3}), \\ x^{(\frac{1}{3},\frac{1}{3},\frac{1}{3})} &= (x_1x_2x_3)^{1/3}. \end{aligned}$$

**Muirhead.** *If $\alpha$, $\beta$ and $x$ are as above, and $\alpha$ majorizes $\beta$, then*

$$x^\alpha \geq x^\beta.$$

*The equality holds only when $\alpha = \beta$ or all $x_i$ are equal.*

Although an ad-hoc proof is possible (search the web), the best conceptual proof is given in Steele's book. In high-school competition circles, this is often called "bunching principle," but be aware that this inequality is not as widely known as others. Ultimately, it follows from a judicious application of weighted AM-GM, so it may be a good advice to convert the solutions accordingly.

Another remark is this related inequality below.

**Majorization.** *If $\alpha_i, \beta_i \in I$, $\alpha$ majorizes $\beta$, and $f$ is convex on $I$, then*

$$f(\alpha_1) + \cdots + f(\alpha_n) \geq f(\beta_1) + \cdots + f(\beta_n).$$

To prove this, you really need to read Chapter 13 of Steele's book.

Here is another very particular, but popular, homogeneous inequality.

**Schur.** *Assume that $r > 0$, $x, y, z \geq 0$, then*

$$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \geq 0,$$

*with equality if $x, y, z$ are equal, or two are equal and the third is zero.*

To prove it, note that we can assume $x \geq y \geq z$ and then rewrite the inequality as

$$(x-y)[x^r(x-z) - y^r(y-z)] + z^r(x-z)(y-z) \geq 0.$$

Note that (due to the equality condition), this cannot follow from Muirhead. In fact, for $r = 1$, it says

$$2x^{(2,1,0)} \le x^{(3,0,0)} + y^{(1,1,1)},$$

and $(2, 1, 0)$ of course majorizes $(1, 1, 1)$ but not $(3, 0, 0)$. So this inequality can sometimes be used to eliminate a nuisance term before applying Muirhead.

Let's do an example, from IMO. Let

$$F = \frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)}.$$

Find the range of $F$ under the constraints $a, b, c > 0$ and $abc = 1$.

This is easier if we take $x = 1/a$, $y = 1/b$, $z = 1/c$ to get

$$F = \frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}.$$

Then it is clear that $F$ could be made arbitrarily large, so we need to check for which $a$ we can show $F \ge a$. This not homogeneous, so we homogenize by making it

$$F \ge a(xyz)^{1/3}.$$

Then we can also drop the constraint. (Incidentally, you *could* use Lagrange multipliers, but they would lead to ungainly mess.) Now multiply out to get

$$3x^{(2,0,0)} \ge a \cdot 2x^{(4/3,1/3,1/3)}.$$

So we can take $a = 3/2$, which is obviously achieved when $a = b = c = 1$. The range is $[3/2, \infty)$.

The problems below have mainly been chosen from competitions. Most of them have more than one solution (which is not surprising, as the above equalities are to a large extent related). Try to find the simplest. You should not forget calculus — you should however be warned that, first, most competition problems are designed to be be computationally unpleasant when solved by Lagrange multipliers and, second, that you do need to be *completely rigorous* and calculus only gives you *necessary* conditions for an extremum in the *interior* of a region.

1. Assume $a, b, c > 0$. Show that

$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \le \frac{1}{abc}.$$

2. Compute the minimum of

$$x_1 + \frac{1}{2}x_2^2 + \cdots + \frac{1}{n}x_n^2,$$

where $x_1, \ldots, x_n \ge 0$ and

$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} = n.$$

3. Suppose $a, b, c, d, e$ are real numbers which satisfy

$$a+b+c+d+e=8, \quad a^2+b^2+c^2+d^2+e^2=16.$$

Determine the maximal value of $e$.

4. Determine the maximum of

$$F = \frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c)$$

for $a, b, c \in [0,1]$.

5. Prove that

$$\frac{a_n}{a_1} + \frac{a_1}{a_2} + \cdots + \frac{a_{n-1}}{a_n} \geq n$$

for $a_1, \ldots, a_n > 0$, with equality only when $a_1 = \cdots = a_n$.

6. (*) Show that

$$(ab+bc+ca)((a+b)^{-2} + (b+c)^{-2} + (c+a)^{-2}) \geq 2.25$$

when $a, b, c > 0$.

7. (*) Suppose that $x_1, x_2, x_3$ are three points on a circle of radius 1. Find the minimum of

$$\frac{1}{||x_1 - x_2||} + \frac{1}{||x_2 - x_3||} + \frac{1}{||x_3 - x_1||}.$$

8. Find all polynomials with coefficients ±1 which only have real roots.

9. (*) Suppose that $f(x,y)$ is a continuous real-valued function on the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$. Show that

$$\begin{aligned} &\int_0^1 \left( \int_0^1 f(x,y)dx \right)^2 dy + \int_0^1 \left( \int_0^1 f(x,y)dy \right)^2 dx \\ &\leq \left( \int_0^1 \int_0^1 f(x,y)dx\,dy \right)^2 + \int_0^1 \int_0^1 (f(x,y))^2\,dx\,dy. \end{aligned}$$

10. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq [(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)]^{1/n}.$$

11. Let $\alpha$, $\beta$, $\gamma$ be angles of an acute triangle. What are the possible values of $\cos\alpha + \cos\beta + \cos\gamma$?

12. Let $x_1, x_2, ..., x_n \in [0,2)$ have sum 1. Prove that

$$\frac{x_1}{2-x_1} + \frac{x_2}{2-x_2} + ... + \frac{x_n}{2-x_n} \geq \frac{n}{2n-1} \ .$$

13. (*) A slot machine offers you the following game. Three symbols appear in a row on your screen. Each symbol is either a cherry, a lemon, or a pear. Assume that each symbol is chosen independently, but the probabilities are unknown. You win if either all three symbols are the same or if all are different. You hear somebody complain that without knowing the probabilities you have no information about your winning probability. Counter that argument by giving the precise lower bound.

14. Assume that $a_{ij}$, $b_{jk}$, $c_{ki}$, $1 \leq i, j, k \leq n$ are nonnegative numbers. Prove:

$$\sum_{i,j,k} a_{ij}^{1/2} b_{jk}^{1/2} c_{ki}^{1/2} \leq \left(\sum_{i,j} a_{ij}\right)^{1/2} \left(\sum_{j,k} b_{jk}\right)^{1/2} \left(\sum_{k,i} c_{ki}\right)^{1/2} .$$

Assume that $A \subset \mathbb{Z}^3$, and $A_x$, $A_y$, and $A_z$, respectively, are projections onto coordinate planes, $yz$-plane, $xz$-plane, and $xy$-plane, respectively. Prove that $|A| \leq |A_x|^{1/2}|A_y|^{1/2}|A_z|^{1/2}$.

15. Assume $x_1, \ldots, x_n \geq 0$ and that $x_1 \cdots x_n \geq 1$. Find the minimum of

$$\prod_{i=1}^{n} (1 + x_i)$$

under these conditions.

16. Take a box with sides $a, b, c > 0$ and surface area $S$. Show that among all such boxes, the cube has the largest volume.

17. Show that for positive $x, y, z$,

$$\frac{1}{18}(x+y+z)^3 \leq \frac{x^4}{y+z} + \frac{y^4}{x+z} + \frac{z^4}{x+y}.$$

18. Assume that $x_1, \ldots, x_n > 0$, and $max\{x_1, \ldots, x_n\} \leq (x_1 + \cdots + x_n)/k$ for some $1 \leq k \leq n$. Show that

$$\sum_{i=1}^{n} \frac{1}{1+x_i} \leq n - k + \frac{k^2}{k + x_1 + \cdots + x_n}.$$

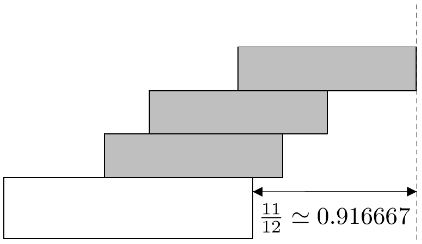
19. A person's birthday occurs on a day $i$ with probability $p_i$, where $i = 1, \ldots, k$. (Of course, $p_1 + \cdots + p_k = 1$.) Assume independent assignment of birthdays among different people. In

a room with $n$ people, let $P_n = P_n(p_1, \dots, p_k)$ be the probability that no two persons share a birthday. Show that this probability is maximized when $p_i = 1/k$ for all $k$.

20. Let $x_i \in (-1, 1)$, $i = 0, \dots, n$, and $x_0 + \cdots + x_n \geq n - 1$. Prove that

$$\prod_{i=0}^{n} \frac{1 + x_i}{1 - x_i} \geq n^{n+1}.$$

21. You have $n$ identical bricks of unit length. Put the first brick on the edge of a table, perpendicular to the edge, so that it overhangs but does not fall. Put the second brick on top of the first (precisely aligned with the first, except for the overhang in the same direction) so that the entire structure is stable. Put the third brink on top of the second in the same way, etc. How far over the edge of the table can the stable structure extend? The optimal solution with three bricks is below:



*Note*. This problem is at least 150 years old. It received some very recent attention as it was shown that one can dramatically increase the overhang (to about $n^{1/3}$) if some bricks may be used as counterweigths, that is, if the restriction that a brick can touch at most one brick below it is removed.