Problem Set 7: Monotonicity

This method is related to both the variational method and the invariants from previous problem sets. Suppose you perform a series of steps and you observe that no matter what step you take, you decrease a certain quantity (or perhaps increase another one). Such quantity, sometimes called a semiinvariant or monoinvariant, now must be a real-valued function on the objects and inspiration is often required to find it. In most cases, a useful monotone quantity is not unique! (In the context of differential equations such functions are often called Lyapunov functions.) The example below is famous, due, as many such problems, to John Conway, and is not easy to solve on the spot.

Each integer point on the plane on or below the $x$-axis is occupied by a peg. At each time, a peg can jump over a horizontal or vertical neighbor, provided it is occupied, and onto the next square, provided it is unoccupied. The jumped peg is then removed. Can you get a peg arbitrarily far above the $x$-axis? (This is the original formulation, some versions allow also diagonal jumps.)

You should first try to get a peg onto line $y = 4$. This is not a very easy problem! Configurations with minimal numbers of pegs for lines $y = 1, 2, 3, 4$ (together with the target square, marked by $\times$) are given below.

We will now show that no higher position is possible. To show this is it enough to show that we can never reach $(0, y)$ for $y \geq 5$, with this point being the first point ever above the line $y = 4$. The vague idea is to define a function that is the sum of weights over all integer sites and “assigns high weights to high points on the $y$-axis.” There are of course many such functions, but one with exponential weights is a natural one to try, especially since we have to make sure the final sum is finite. So we define the weights

$$w(x, y) = r^{y-|x|}.$$  

Here, $r > 1$ is a parameter which we will choose later. Now the function $F$ from all possible distributions of pegs to $\mathbb{R}$ is given by

$$F = \sum_{(x, y) \text{ occupied by a peg}} w(x, y).$$

Let’s start by showing that this is a finite function at the initial configuration. It will follow that it is finite at every possible configuration, because after $s$ steps $F$ is at most $r^{2s}$ times what
it was at the beginning. At the beginning, 

\[ F_0 = \sum_{j=0}^{\infty} r^{-j} + 2 \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} r^{-i-j} = \frac{r}{r-1} + 2 \sum_{i=1}^{r-1} + 2 \frac{r}{(r-1)^2} = \frac{r^2 + r}{(r-1)^2}, \]

which is obviously finite.

Assume that a peg with weight \(w\) jumps. Then at best, when the jump is vertically upward, \(F\) gains \(wr^2\) and loses \(w + wr\). As long as \(r\) is at most the largest root of \(r^2 = r + 1\), \(F\) can not increase at any step. Therefore we choose \(r = (1 + \sqrt{5})/2\), the golden ratio. Now the weight of the peg at \((0,5)\) by itself is \(r^{1/2}\) which happens to be equal to the initial value \(F_0\). Check this! But of course one has to have at least one more peg somewhere so the value at this time has to be larger than \(F_0\), a contradiction.

1. Start with \(x_0 = a > 0\), and let \(x_n\) be the sequence of real numbers defined by

\[ x_{n+1} = \begin{cases} 
 x_n^2 + 1, & \text{if } n \text{ even,} \\
 \sqrt{x_n} - 1, & \text{if } n \text{ is odd.}
\end{cases} \]

Determine the limit points of this sequence.

2.(*) Suppose you have a pair of nonnegative integers \((a, b)\). Replace it with a pair \((|2a - b|, |2b - a|)\), then repeat. Under what conditions does at least one of the numbers eventually become larger than a million?

3.(*) To each vertex of a regular \(n\)-gon \((n \geq 3)\) an integer is assigned, so that the sum of all \(n\) numbers is positive. If three consecutive vertices are assigned the numbers \(x, y, z\) respectively, and \(y < 0\), then the following operation is allowed: \(x, y, z\) are replaced by \(x + y, -y, z + y\) respectively. (Note that this does not change the sum of all \(n\) numbers.) Three such vertices are chosen, and this operation is performed, repeatedly as long as at least one of the \(n\) numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Note. When I first tried to solve this problem, I overlooked the positive sum assumption. The conclusion is not true without it. Rule number 1: read the problem carefully!

4. A number, say \(n\), of tennis players play a tournament according to the following rules. First they play a round robin tournament, in which each pair of players plays a single game. A computer then puts them in a random order. In an adjustment step, the computer chooses (again at random) any two adjacent players in the order \((n - 1)\) choices; if the player ahead in the order beat the player immediately behind in the order, the order of the two is reversed (otherwise the adjustment step does nothing). The computer repeatedly performs adjustment steps until no more order reversals are possible. Must this procedure always end in finitely many steps?

Note that the procedure may not end if adjustment steps between non-adjacent players are allowed. An example are three players A, B, C, such that A beat B, B beat C, and C beat A.
5. An $n \times n$ chessboard initially contains $b$ pegs. At every step afterwards, you (simultaneously, say) add a peg to every square which has at least two among its neighbors occupied by a peg. The neighbors are those squares which share an edge, so a square can have at most four neighbors. You never remove a peg. Assume that this procedure completely fills the chessboard. Show that $b \geq n$.

6. Cards in a deck are numbered $1, \ldots, n$. Put them in a random order in a stack on the table. Note the top card. If its number is $k$, reverse the order of top $k$ cards. Repeat. Note that once the top card is 1, no more reversals are done. Prove that this eventually happens.

7. (*) You have died and went to Purgatory. You are given a box of billiard balls, each ball having a positive integer (1 or 2 or \ldots) written on it. The box has infinite capacity, although you are assured it contains only finitely many balls. On day 1, you reach blindly into the box and take out a ball. The number $i_1$ on it is noted by the authorities, and replaced by a ball of some lower denomination, unless $i_1 = 1$ in which case no ball is put into the box and the number of balls decreases by 1. This repeats on day 2, except that now 2 balls of some lower denomination, if any, are placed into the box. And so on. For example, if you pull a ball numbered 17 on day 100, a hundred new balls are put into the basket, and they may have any denominations from 16 down to 1. If you pull a 1–ball on the same day, your box will contain one fewer ball. When the box is empty, your penance is over.

Should you complain that your time in Purgatory may be endless?

8. Start with $n$ integers $x_0, \ldots, x_{n-1}$, transform them into another $n$ numbers $|x_{i+1} - x_i|$, $i = 0, \ldots, n - 1$ (with $x_n$ interpreted as $x_0$), then iterate. Assume that $n$ is a power of 2. Show that all the numbers eventually become 0.

9. (*) There is a row of $n$ containers of infinite capacity, which together contain $m$ coins. Each time, you choose a container. If the container contains at least two coins and is not at either end of the row, you take two coins out and put one into each of the neighboring containers. Show that this activity will eventually stop.