Math 21A, Fall 2015.
Oct. 21, 2015.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, first name first): ________________________________

NAME(sign): ________________________________

ID#: ________________________________

Instructions: Each of the 5 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 6 pages (including this one) with 5 problems.

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1. In both parts of this problem, consider the function

\[ f(x) = \begin{cases} 
  x & \text{if } x \leq -2 \\
 ax^2 + b & \text{if } -2 < x < 1 \\
 -2x + 3 & \text{if } x \geq 1 
\end{cases} \]

(a) Determine \(a\) and \(b\) so that the function \(y = f(x)\) is continuous for all \(x\). Then sketch its graph and determine its range.

\[
\begin{align*}
\text{Cont. at } -2 & : -2 = 4a + b \\
\text{Cont. at } 1 & : 1 = a + b \\
\end{align*}
\]

\[ 3a = -3 \quad \Rightarrow \quad a = -1 \]

\[ b = 1 - a = 2 \]

\[ \text{Range: } (-\infty, 2] \]

(b) Assume values of \(a\) and \(b\) determined in (a). For which \(x\) is the function \(y = f(x)\) differentiable?

\[
\begin{align*}
\text{Diff. at } -2 & : \begin{align*}
\text{Left der.} & = 1 \\
\text{Right der.} & = -2x \bigg|_{x=-2} = 4 \quad \text{Not diff.}
\end{align*} \\
\text{Diff. at } 1 & : \begin{align*}
\text{Left der.} & = -2x \bigg|_{x=1} = -2 \\
\text{Right der.} & = -2 \quad \text{diff.}
\end{align*}
\end{align*}
\]

Answer: \(y = f(x)\) is differentiable everywhere but at \(x = -2\).
2. Consider the function \( f(x) = \frac{3(x^2 - 4x)}{x^2 - 4} \). Determine the domain, intercepts, and vertical and horizontal asymptotes. Determine also any points where the graph of \( y = f(x) \) intersects its horizontal asymptote. Then sketch the graph of this function on which all obtained points and asymptotes are clearly marked.

**Domain:** all \( x \) but \( x = 2, x = -2 \).

**Intercepts:** \( y\text{-int.}: (0,0), \ x\text{-int.}: (0,0), (4,0) \)

**Horizontal asymptote:** \( \lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \frac{3x^2}{x^2} = 3 \)

\[ y = 3 \]

**Interactions with h.a.:** \( \frac{3(x^2 - 4x)}{x^2 - 4} = 3 \cdot \frac{x^2 - 4x}{x^2 - 4} \)

\[ (1, 3) \]

**Vertical asymptotes:** \( x = 2 \)

\[ \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{(x-2)(x+2)}{3(x-2)(x+2)} \]

\[ = \infty \]

\[ \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{3x(x-4)}{x^2 - 4} \]

\[ \approx 4 \]

\[ \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{3x(x-4)}{x^2 - 4} \]

\[ = -\infty \]

\[ \lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \frac{3x(x+4)}{x^2 - 4} \]

\[ \approx 4 \]

\[ \lim_{x \to -2} f(x) = -\infty \]

\[ \lim_{x \to -2^-} f(x) = +\infty \]
3. Compute the following limits. Give each answer as a finite number, $+\infty$, or $-\infty$.

(a) \( \lim_{x \to 0} \frac{x}{\sqrt{x^2 + 1} - \sqrt{x + 1}} \)

\[
= \lim_{x \to 0} \frac{x}{(\sqrt{x^2 + 1})^2 - (\sqrt{x + 1})^2} \\
= \lim_{x \to 0} \frac{x}{x(\sqrt{x} - 1)} \\
= \frac{2}{-1} = -2
\]

(b) \( \lim_{x \to 0} \frac{x + x^2 + \sin(3x)}{x - x^2 - \sin(3x)} \)

\[
= \lim_{x \to 0} \frac{1 + x + \frac{\sin 3x}{3x}}{1 - x - \frac{\sin 3x}{3x}} \\
= \frac{1 + 3}{1 - 3} = -\frac{2}{2} = -1
\]
4. In all parts of this problem, \( f(x) = \frac{x^2 + 3}{x + 3} \).

(a) Determine \( L = \lim_{x \to 1} f(x) \).

\[
L = \frac{1 + 3}{1 + 3} = \frac{4}{4} = 1
\]

(b) For a given \( \epsilon > 0 \), determine a \( \delta > 0 \) so that \(|x - 1| < \delta\) will guarantee that \(|f(x) - L| < \epsilon\).

\[
\left| f(x) - 1 \right| < \epsilon
\]

\[
\left| \frac{x^2 + 3}{x + 2} - 1 \right| < \epsilon
\]

\[
\left| \frac{x^2 + x - x - 3}{x + 3} \right| < \epsilon
\]

\[
\frac{|x - 1| \cdot |x|}{|x + 3|} < \epsilon
\]

Assume \( |x - 1| < 1 \); then \( 0 < x < 2 \)

\[3 < x + 3 < 5\]

The above is then implied by

\[
\frac{|x - 1| \cdot 2}{3} < \epsilon, \quad |x - 1| < \frac{3}{2} \epsilon.
\]

Take \( \delta = \min \left( \frac{3}{2} \epsilon, 1 \right) \).
5. In all parts of this problem, \( f(x) = x + \frac{2}{x} - 1 \).

(a) A line is tangent to the graph of \( y = f(x) \) and goes through the origin (i.e., through the point \((0,0))\). Determine the equation of this line (in the slope-intercept form).

\[
\frac{d}{dx} f(x) = 1 - \frac{2}{x^2}
\]

Tangent at a pt. \((a, f(a))\):

\[
y - \left(a + \frac{2}{a} - 1\right) = \left(1 - \frac{2}{a^2}\right)(x-a)
\]

Plug \( m = y = 0 \):

\[-a - \frac{2}{a} + 1 = -a + \frac{2}{a}
\]

\[
\frac{4}{a} = 1, \quad a = 4
\]

Slope: \(1 - \frac{2}{16} = \frac{7}{8}\)

Line: \( y = \frac{7}{8}x \).

(b) Let \( g(x) = x^4 \). Do the graphs of \( y = f(x) \) and \( y = g(x) \) intersect for some \( x \geq 1 \)? Clearly explain your answer.

Let \( h(x) = f(x) - g(x) \). Thus \( h(x) \) is continuous function for \( x \geq 1 \).

\[
h(1) = f(1) - g(1) = 1 + 2 - 1 - 1 = 1 > 0
\]

\[
h(2) = f(2) - g(2) = 2 + 1 - 1 - 16 = -14 < 0
\]

By IVT, there is an \( x \) between 1 and 2 such that \( h(x) = 0 \), i.e., \( f(x) = g(x) \).