
MIDTERM EXAM 2

NAME(print in CAPITAL letters, first name first):

NAME(sign): KEY

ID#:

Instructions: Each of the four problems is worth 25 points. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor in determining credit. Calculators, books or notes are not allowed.

Make sure that you have a total of 5 pages (including this one) with 4 problems. Read through the entire exam before beginning to work.

1
2
3
4
TOTAL

\[
\begin{align*}
\sin A \sin B &= \frac{1}{2} (\cos(A - B) - \cos(A + B)) \\
\sin A \cos B &= \frac{1}{2} (\sin(A - B) + \sin(A + B)) \\
\cos A \cos B &= \frac{1}{2} (\cos(A - B) + \cos(A + B)) \\
\sin^2 A &= \frac{1}{2} (1 - \cos(2A)), \quad \cos^2 A = \frac{1}{2} (1 + \cos(2A))
\end{align*}
\]
1. You have 10 pounds of radioactive isotope A and 40 pounds of radioactive isotope B. Half-life of A is 5 years, half-life of B is 2 years. 
   (a) How much time must elapse before only 1 pound of A remains? (Give the final answer as a decimal number using the approximation $\frac{\ln 5}{\ln 2} \approx 2.3$.)

   \[
   \begin{align*}
   A: \quad P_1(t) &= P_1(0) e^{k_1 t} = 10 e^{k_1 t} \\
   B: \quad P_2(t) &= P_2(0) e^{k_2 t} = 40 e^{k_2 t}
   \end{align*}
   \]

   \[
   k_1 = -\ln 2, \quad k_2 = -\frac{1}{2} \ln 2
   \]

   Time t until 1 lb of A: \[ P_1(t) = 1 \]

   \[
   \begin{align*}
   10 e^{k_1 t} &= 1 \\
   \ln 10 + k_1 t &= 0
   \end{align*}
   \]

   \[
   t = \frac{-\ln 10}{k_1} = \frac{\ln 10}{\ln 2} = \frac{\ln 5}{\ln 2} + 1
   \]

   \[= 5 \cdot (3.3) = 16.5 \text{ (years)}
   \]

(b) When are quantities of A and B equal? Express your (exact) result as a simple fraction of years.

\[
\begin{align*}
P_1(t) &= P_2(t) \\
10 e^{k_1 t} &= 40 e^{k_2 t} \\
e^{k_1 t} &= 4 e^{k_2 t} \\
k_1 t &= \ln 4 + k_2 t
\end{align*}
\]

\[
t = \frac{\ln 4}{k_1 - k_2} = \frac{2 \ln 2}{-\frac{1}{2} \ln 2 + \frac{1}{2} \ln 2} = \frac{2}{3/10} = \frac{20}{3} \text{ (years)}
\]
2. Compute the following indefinite integrals.

(a) \[ \int \frac{1}{(9 + x^2)^{3/2}} \, dx \]

\[ x = 3 \tan t \]
\[ dx = 3 \sec^2 t \, dt \]
\[ 9 + x^2 = 9 (1 + \tan^2 t) = 9 \sec^2 t \]

\[ = \int \frac{3 \sec^2 t}{27 \sec^2 t} \, dt = \frac{1}{9} \int \cos t \, dt \]
\[ = \frac{1}{9} \sin t + C = \frac{1}{9} \tan t \cos t + C = \frac{1}{9} \tan t \cdot \frac{1}{\sqrt{\tan^2 t + 1}} + C \]
\[ = \frac{1}{9} \cdot \frac{x}{3} \cdot \frac{1}{\sqrt{(x^2/9) + 1}} + C = \frac{1}{9} \cdot \frac{x}{\sqrt{x^2 + 9}} + C \]

(b) \[ \int \frac{1}{x + 2x^{2/3} - 3x^{1/3}} \, dx \]

\[ x = t^3 \quad dx = 3t^2 \, dt \]

\[ = \int \frac{3t^2 \, dt}{t^3 + 2t^2 - 3t} = 3 \int \frac{t \, dt}{t^2 + 2t - 3} \]
\[ \frac{t}{t^2 + 2t - 3} = \frac{t}{(t+3)(t-1)} = \frac{A}{t+3} + \frac{B}{t-1} \]
\[ t = A(t-1) + B(t+3) \]
\[ t = 1: \quad B = \frac{1}{4} \]
\[ t = -3: \quad A = \frac{3}{4} \]

\[ = 3 \cdot \frac{3}{4} \ln |t+3| + 3 \cdot \frac{1}{4} \ln |t-1| + C \]
\[ = \frac{9}{4} \ln |x^{1/2} + 3| + \frac{3}{4} \ln |x^{1/3} - 1| + C \]
3. The region $R$ lies between $x = 0$ and $x = \pi$ and is bounded by the graphs of $y = \sin(x/2)$ and $y = 0$.

(a) Rotate $R$ around the $y$-axis and compute the volume of the resulting solid.

\[
\int_0^\pi x \sin \frac{x}{2} \, dx
\]

\[
= 2\pi \left[ x \cdot (-2\cos \frac{x}{2}) \right]_0^\pi + 2 \int_0^\pi \cos \frac{x}{2} \, dx
\]

\[
= 4\pi \left[ 2\sin \frac{x}{2} \right]_0^\pi = \frac{8\pi}{\pi}
\]

(b) Rotate the region around the line $y = -2$ and compute the volume of the resulting solid.

\[
\int_0^\pi \left[ (\sin \frac{x}{2} + 2)^2 - 4 \right] \, dx
\]

\[
= \int_0^\pi \left[ \sin^2 \frac{x}{2} + 4\sin \frac{x}{2} \right] \, dx
\]

\[
= \int_0^\pi \left[ \frac{1}{2} (1 - \cos x) + 4\sin \frac{x}{2} \right] \, dx
\]

\[
= \left[ \frac{1}{2} x - \frac{1}{2} \sin x - 8\cos \frac{x}{2} \right]_0^\pi
\]

\[
= \int_0^\pi \left[ \frac{\pi}{2} + 8 \right] = \frac{\pi^2}{2} + 8\pi
\]
4. The region $R$ is bounded by curves $y = (x - 4)^2$ and $y = 2x$ and has constant density 1. Set up the definite integrals for coordinates of its centroid. Do not compute the integrals, but explain briefly how you would compute them.

\[
(x-4)^2 = 2x \\
x^2 - 8x + 16 = 2x \\
x^2 - 10x + 16 = 0 \\
(x - 8)(x - 2) = 0 \\
x = 2, 8
\]

\[
\text{Area of } R = \int_{2}^{8} (2x - (x-4)^2) \, dx
\]

\[
\bar{x} = \frac{\int_{2}^{8} x \cdot (2x - (x-4)^2) \, dx}{\text{Area of } R}
\]

\[
\bar{y} = \frac{\frac{1}{2} \int_{2}^{8} ((2x)^2 - (x-4)^4) \, dx}{\text{Area of } R}
\]

Multiply out & use the power rule.