
Discussion Problems 9 (Thu., Mar. 15)

1. Determine whether the following integrals converge or diverge.

(a) \( \int_{0}^{\infty} \frac{e^{-x}}{1 + x^2} \, dx \)  
(b) \( \int_{1}^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} \, dx \)  
(c) \( \int_{0}^{3} \frac{1}{(x - 3)^2} \, dx \)  
(d) \( \int_{0}^{\infty} \frac{x^{9/2} - x^{7/2}}{x^6 + x^3 + x^{3/2} + 1} \, dx \)  
(e) \( \int_{0}^{5} \frac{1}{\sqrt{25 - x^2}} \, dx \)  
(f) \( \int_{1}^{e} \frac{1}{x \ln x} \, dx \)  
(g) \( \int_{10}^{\infty} \frac{x^5 \arctan x}{x^7 + 1} \, dx \)  
(h) \( \int_{10}^{\infty} \frac{x^5 \ln x}{x^7 + 1} \, dx \)

2. A particle moves on a curve \( x = t + \cos t, \ y = 2t - \sin t \). (a) Find the particle’s velocity (which is the derivative \( ds/dt \) of the arc length \( s \)) at time \( t = \pi/2 \). (b) Find the tangent to the curve at time \( t = 0 \). (c) Sketch the curve roughly using first derivatives. (d) Find the area under the curve for \( 0 \leq t \leq \pi/2 \). (e) Rotate the region in (d) around the \( y \)-axis. Set up the integral for the volume of the resulting solid. (f) Set up the integral for the arc length of this curve for \( 0 \leq t \leq \pi/2 \).

3. Consider the equation \( r = 1 + 2 \sin \theta \) in polar coordinates. Restrict yourself to \( \theta \) for which \( 1 + 2 \sin \theta \geq 0 \). (a) Sketch the graph of the resulting curve. (b) Compute the point on this curve with maximal distance from the origin. (c) Compute the point on this curve with maximal distance from the \( y \)-axis. (d) Compute the area enclosed by the curve. (e) Compute the area of the intersection between the region enclosed by the curve and the interior of the circle \( r = 3/2 \). (f) Set up the integral for the arc length of this curve.
We had come on a small black ribbon of pathway. In the middle of it, clearly marked on the sodden soil, was the track of a bicycle.

“Hurrah!” I cried. “We have it.”

But Holmes was shaking his head, and his face was puzzled and expectant rather than joyous.

“A bicycle, certainly, but not the bicycle” said he. “I am familiar with forty-two different impressions left by tires. This, as you perceive, is a Dunlop, with a patch upon the outer cover. Heideggers tires were Palmers, leaving longitudinal stripes. Aveling, the mathematical master, was sure upon the point. Therefore, it is not Heideggers track.”

“The boys, then?”

“Possibly, if we could prove a bicycle to have been in his possession. But this we have utterly failed to do. This track, as you perceive, was made by a rider who was going from the direction of the school.”

“Or towards it?”

“No, no, my dear Watson. The more deeply sunk impression is, of course, the hind wheel, upon which the weight rests. You perceive several places where it has passed across and obliterated the more shallow mark of the front one. It was undoubtedly heading away from the school. It may or may not be connected with our inquiry, but we will follow it backwards before we go any farther.”

We did so, and at the end of a few hundred yards lost the tracks as we emerged from the boggy portion of the moor. Following the path backwards, we picked out another spot, where a spring trickled across it. Here, once again, was the mark of the bicycle, though nearly obliterated by the hoofs of cows. After that there was no sign, but the path ran right on into Ragged Shaw, the wood which backed on to the school. From this wood the cycle must have emerged. Holmes sat down on a boulder and rested his chin in his hands. I had smoked two cigarettes before he moved.

“Well, well,” said he, at last. “It is, of course, possible that a cunning man might change the tires of his bicycle in order to leave unfamiliar tracks. A criminal who was capable of such a thought is a man whom I should be proud to do business with. We will leave this question undecided and hark back to our morass again, for we have left a good deal unexplored.”

We continued our systematic survey of the edge of the sodden portion of the moor, and soon our perseverance was gloriously rewarded. Right across the lower part of the bog lay a miry path. Holmes gave a cry of delight as he approached it. An impression like a fine bundle of telegraph wires ran down the centre of it. It was the Palmer tires.

“Here is Herr Heidegger, sure enough!” cried Holmes, exultantly. “My reasoning seems to have been pretty sound, Watson.”