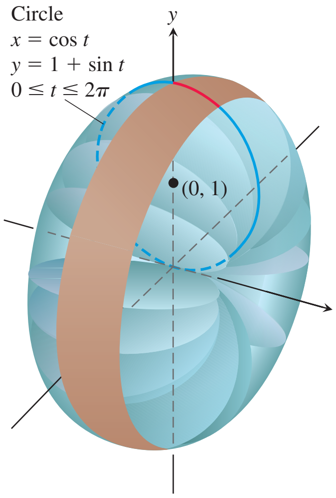


FIGURE 11.18 In Example 9 we calculate the area of the surface of revolution swept out by this parametrized curve.

EXAMPLE 9 The standard parametrization of the circle of radius 1 centered at the point $(0, 1)$ in the xy-plane is

$$x = \cos t, \qquad y = 1 + \sin t, \qquad 0 \le t \le 2\pi.$$

Use this parametrization to find the area of the surface swept out by revolving the circle about the x-axis (Figure 11.18).

Solution We evaluate the formula

$$\begin{aligned} S &= \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{Eq. (5) for revolution about the } x\text{-axis; } y = 1 + \sin t \ge 0 \\ &= \int_0^{2\pi} 2\pi(1 + \sin t)\underbrace{\sqrt{(-\sin t)^2 + (\cos t)^2}}_{1}\, dt \\ &= 2\pi \int_0^{2\pi} (1 + \sin t)\, dt \\ &= 2\pi \Big[t - \cos t\Big]_0^{2\pi} = 4\pi^2. \end{aligned}$$

Exercises 11.2

Tangents to Parametrized Curves

In Exercises 1–14, find an equation for the line tangent to the curve at the point defined by the given value of t. Also, find the value of d^2y/dx^2 at this point.

1. $x = 2\cos t, \quad y = 2\sin t, \quad t = \pi/4$
2. $x = \sin 2\pi t, \quad y = \cos 2\pi t, \quad t = -1/6$
3. $x = 4\sin t, \quad y = 2\cos t, \quad t = \pi/4$
4. $x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = 2\pi/3$
5. $x = t, \quad y = \sqrt{t}, \quad t = 1/4$
6. $x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\pi/4$
7. $x = \sec t, \quad y = \tan t, \quad t = \pi/6$
8. $x = -\sqrt{t+1}, \quad y = \sqrt{3t}, \quad t = 3$
9. $x = 2t^2 + 3, \quad y = t^4, \quad t = -1$
10. $x = 1/t, \quad y = -2 + \ln t, \quad t = 1$
11. $x = t - \sin t, \quad y = 1 - \cos t, \quad t = \pi/3$
12. $x = \cos t, \quad y = 1 + \sin t, \quad t = \pi/2$
13. $x = \dfrac{1}{t+1}, \quad y = \dfrac{t}{t-1}, \quad t = 2$
14. $x = t + e^t, \quad y = 1 - e^t, \quad t = 0$

Implicitly Defined Parametrizations

Assuming that the equations in Exercises 15–20 define x and y implicitly as differentiable functions $x = f(t)$, $y = g(t)$, find the slope of the curve $x = f(t)$, $y = g(t)$ at the given value of t.

15. $x^3 + 2t^2 = 9, \quad 2y^3 - 3t^2 = 4, \quad t = 2$
16. $x = \sqrt{5 - \sqrt{t}}, \quad y(t-1) = \sqrt{t}, \quad t = 4$
17. $x + 2x^{3/2} = t^2 + t, \quad y\sqrt{t+1} + 2t\sqrt{y} = 4, \quad t = 0$
18. $x\sin t + 2x = t, \quad t\sin t - 2t = y, \quad t = \pi$
19. $x = t^3 + t, \quad y + 2t^3 = 2x + t^2, \quad t = 1$
20. $t = \ln(x - t), \quad y = te^t, \quad t = 0$

Area

21. Find the area under one arch of the cycloid
$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$
22. Find the area enclosed by the y-axis and the curve
$$x = t - t^2, \quad y = 1 + e^{-t}.$$
23. Find the area enclosed by the ellipse
$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi.$$
24. Find the area under $y = x^3$ over $[0, 1]$ using the following parametrizations.
   a. $x = t^2, \quad y = t^6$ **b.** $x = t^3, \quad y = t^9$

Lengths of Curves

Find the lengths of the curves in Exercises 25–30.

25. $x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi$
26. $x = t^3, \quad y = 3t^2/2, \quad 0 \le t \le \sqrt{3}$
27. $x = t^2/2, \quad y = (2t+1)^{3/2}/3, \quad 0 \le t \le 4$
28. $x = (2t+3)^{3/2}/3, \quad y = t + t^2/2, \quad 0 \le t \le 3$
29. $x = 8\cos t + 8t\sin t$, $y = 8\sin t - 8t\cos t$, $0 \le t \le \pi/2$
30. $x = \ln(\sec t + \tan t) - \sin t$, $y = \cos t, \quad 0 \le t \le \pi/3$

Surface Area

Find the areas of the surfaces generated by revolving the curves in Exercises 31–34 about the indicated axes.

31. $x = \cos t, \quad y = 2 + \sin t, \quad 0 \le t \le 2\pi; \quad x$-axis

32. $x = (2/3)t^{3/2}, \quad y = 2\sqrt{t}, \quad 0 \le t \le \sqrt{3}; \quad y$-axis

33. $x = t + \sqrt{2}, \quad y = (t^2/2) + \sqrt{2}t, \quad -\sqrt{2} \le t \le \sqrt{2}; \quad y$-axis

34. $x = \ln(\sec t + \tan t) - \sin t,\ y = \cos t,\ 0 \le t \le \pi/3;\ x$-axis

35. A cone frustum The line segment joining the points (0, 1) and (2, 2) is revolved about the x-axis to generate a frustum of a cone. Find the surface area of the frustum using the parametrization $x = 2t,\ y = t + 1,\ 0 \le t \le 1$. Check your result with the geometry formula: Area $= \pi(r_1 + r_2)$(slant height).

36. A cone The line segment joining the origin to the point (h, r) is revolved about the x-axis to generate a cone of height h and base radius r. Find the cone's surface area with the parametric equations $x = ht,\ y = rt,\ 0 \le t \le 1$. Check your result with the geometry formula: Area $= \pi r$(slant height).

Centroids

37. Find the coordinates of the centroid of the curve

$$x = \cos t + t\sin t, \quad y = \sin t - t\cos t, \quad 0 \le t \le \pi/2.$$

38. Find the coordinates of the centroid of the curve

$$x = e^t\cos t, \quad y = e^t\sin t, \quad 0 \le t \le \pi.$$

39. Find the coordinates of the centroid of the curve

$$x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi.$$

T **40.** Most centroid calculations for curves are done with a calculator or computer that has an integral evaluation program. As a case in point, find, to the nearest hundredth, the coordinates of the centroid of the curve

$$x = t^3, \quad y = 3t^2/2, \quad 0 \le t \le \sqrt{3}.$$

Theory and Examples

41. Length is independent of parametrization To illustrate the fact that the numbers we get for length do not depend on the way we parametrize our curves (except for the mild restrictions preventing doubling back mentioned earlier), calculate the length of the semicircle $y = \sqrt{1 - x^2}$ with these two different parametrizations:

a. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi/2.$

b. $x = \sin \pi t, \quad y = \cos \pi t, \quad -1/2 \le t \le 1/2.$

42. a. Show that the Cartesian formula

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

for the length of the curve $x = g(y),\ c \le y \le d$ (Section 6.3, Equation 4), is a special case of the parametric length formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Use this result to find the length of each curve.

b. $x = y^{3/2}, \quad 0 \le y \le 4/3$

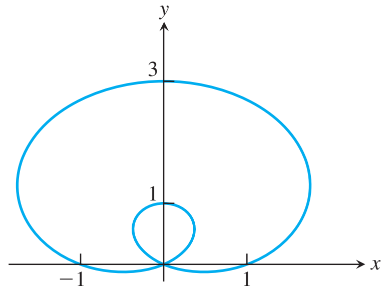
c. $x = \dfrac{3}{2}y^{2/3}, \quad 0 \le y \le 1$

43. The curve with parametric equations

$$x = (1 + 2\sin\theta)\cos\theta, \quad y = (1 + 2\sin\theta)\sin\theta$$

is called a *limaçon* and is shown in the accompanying figure. Find the points (x, y) and the slopes of the tangent lines at these points for

a. $\theta = 0$. **b.** $\theta = \pi/2$. **c.** $\theta = 4\pi/3$.

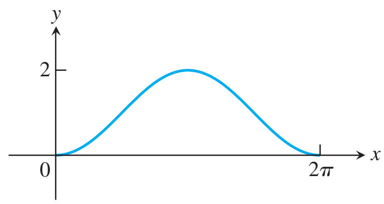


44. The curve with parametric equations

$$x = t, \quad y = 1 - \cos t, \quad 0 \le t \le 2\pi$$

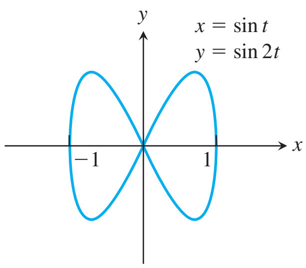
is called a *sinusoid* and is shown in the accompanying figure. Find the point (x, y) where the slope of the tangent line is

a. largest. **b.** smallest.

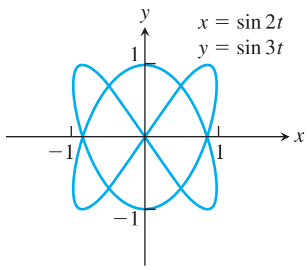


T The curves in Exercises 45 and 46 are called *Bowditch curves* or *Lissajous figures*. In each case, find the point in the interior of the first quadrant where the tangent to the curve is horizontal, and find the equations of the two tangents at the origin.

45.



46.



47. Cycloid

a. Find the length of one arch of the cycloid

$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

b. Find the area of the surface generated by revolving one arch of the cycloid in part (a) about the x-axis for $a = 1$.

48. Volume Find the volume swept out by revolving the region bounded by the x-axis and one arch of the cycloid

$$x = t - \sin t, \quad y = 1 - \cos t$$

about the x-axis.

COMPUTER EXPLORATIONS

In Exercises 49–52, use a CAS to perform the following steps for the given curve over the closed interval.

a. Plot the curve together with the polygonal path approximations for $n = 2, 4, 8$ partition points over the interval. (See Figure 11.15.)